QI Lecture 1

Quantum Mechanics Review
The Quantum State

The Double Slit

Young's "double slit experiment", first done in 1801, is arguably one of the most significant experiments in history. Young used it not only to conclusively verify the wave nature of light, but to actually measure its wavelength for the first time.

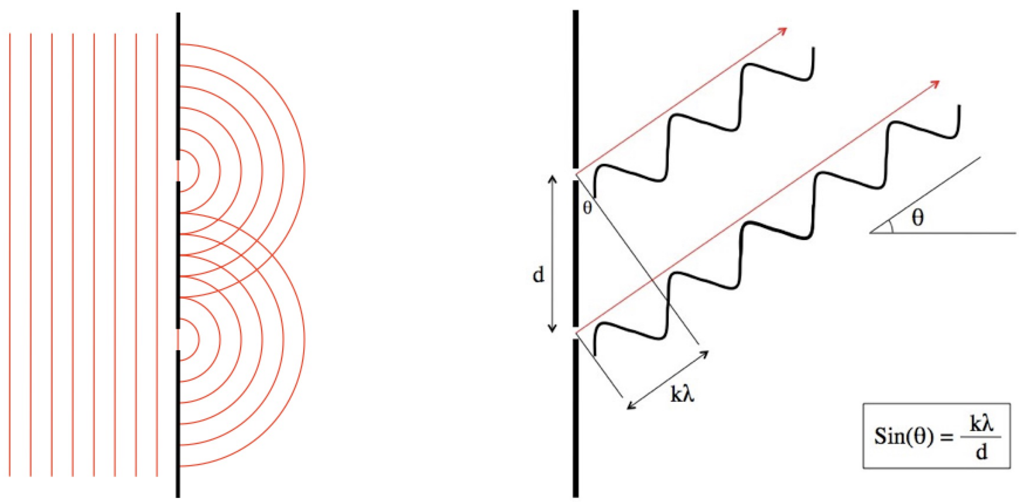

Figure 1: Left: With coherent light, two slits act like synchronized point sources of light waves. Right: Far away (relative to the separation of the slits) we can easily determine where constructive interference will occur.

When light is incident on two thin slits, those slits act like two light sources (this is the "Fresnel Principle" in action). When that light is coherent,[1] the light coming out

[1] Today we use lasers to create coherent light, but when Young originally did this he used a first slit in another barrier before the eponymous two. It also acts like a single light source, and since the distance to the other two slits is fixed, the phase difference between the two slits is fixed. Of course, looking at the interference from the light going through one slit and then two means sitting in a very dark room. Even worse, while laser light is monochromatic, Young only had access to natural light, so he had to deal with overlapping interference fringes of every color.

of the slits is in phase.[2] If the resulting post-slit light is projected onto a screen, we see interference fringes dependent on the separation of the slits and the wavelength of the light.

Although you can certainly calculate what the interference pattern will be in the near field, it's much, much easier to assume that the projection screen is far away relative to the separation between the slits. We can then assume that the angle from each slit, and the distance from each slit, to the relevant point on the screen are equal. This way we can worry about just one angle and assume that the contribution to the amplitude is the same from both slits (since the distance is the same).

Experiment Visible light has wavelengths between 380nm (purple) and 740nm (red), so *about* half a micrometer. The double slits on the slide I have are separated by $d = 0.25mm = 2.50 \times 10^{-4}m$ and my green laser pointer has a wavelength of $\lambda = 532nm = 5.32 \times 10^{-7}m$.

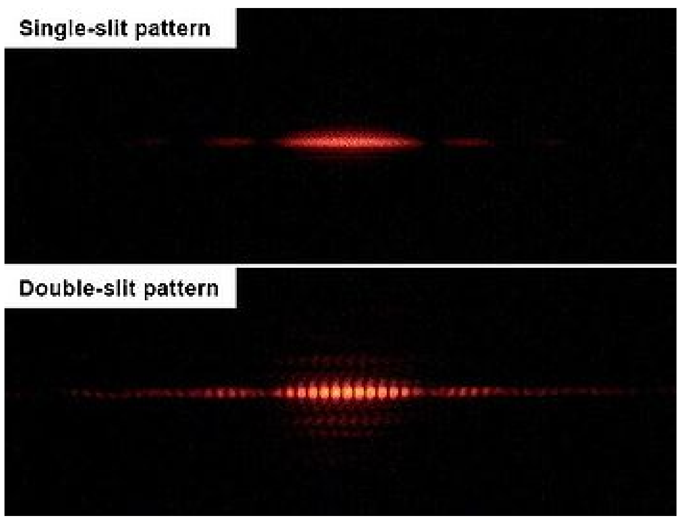

Figure 2: Top: We can see interference from a single thin slit, but it's much clearer... Bottom: with two or more.

Since the wavelength is so much smaller than the separation of the slits, we can be excused for using the small angle approximation for reasonably small values of k.

$$\frac{k\lambda}{d} = \sin(\theta_k) \approx \theta_k$$

and therefore we can expect that the angle between adjacent fringes will be

[2]Or more accurately, the phase difference between them is fixed.

$$\Delta\theta = \theta_{k+1} - \theta_k \approx \frac{(k+1)\lambda}{d} - \frac{k\lambda}{d} = \frac{\lambda}{d} = \frac{5.32 \times 10^{-7} m}{2.50 \times 10^{-4} m} = 2.13 \times 10^{-3}$$

This angle is tiny, which is why we need the length of an entire room. Over a distance of L the separation between the fringes is $L\Delta\theta$. Assuming the wall is about ten meters away, the separation between the dots is

$$L\Delta\theta = (10m)\left(2.13 \times 10^{-3}\right) = 2.13 \times 10^{-2} m$$

or about $2cm$.

In order to get a larger angle we need to decrease the spacing between the slits and to get sharper fringes it turns out that we need more slits.

Experiment Using N slits, evenly spaced d apart, and applying the same approximation (that the screen is very far away compared to the distance between all the slits) we can calculate the sum of amplitudes from every slit. Using the same reasoning behind the double slit experiment above, if the spacing between the slits is d, the wavelength is λ, and the angle deviating from straight ahead is θ, then the phase difference between the contributions of one slit and the next is

$$\Delta = 2\pi \frac{d}{\lambda} \sin(\theta)$$

We're introducing the variable Δ just to save room. Summing up N slits:[3]

$$\begin{aligned}
& \textstyle\sum_{k=0}^{N-1} e^{ik\Delta} \\
= \;& \tfrac{1-e^{iN\Delta}}{1-e^{i\Delta}} \\
= \;& \tfrac{e^{iN\Delta/2}}{e^{i\Delta/2}} \left(\tfrac{e^{-iN\Delta/2}-e^{iN\Delta/2}}{e^{-i\Delta/2}-e^{i\Delta/2}}\right) \\
= \;& e^{i(N-1)\Delta/2} \left(\tfrac{e^{-iN\Delta/2}-e^{iN\Delta/2}}{e^{-i\Delta/2}-e^{i\Delta/2}}\right) \\
= \;& e^{i(N-1)\Delta/2} \left(\tfrac{e^{iN\Delta/2}-e^{-iN\Delta/2}}{e^{i\Delta/2}-e^{-i\Delta/2}}\right) \\
= \;& e^{i(N-1)\Delta/2} \left(\tfrac{\frac{e^{iN\Delta/2}-e^{-iN\Delta/2}}{2i}}{\frac{e^{i\Delta/2}-e^{-i\Delta/2}}{2i}}\right) \\
= \;& e^{i(N-1)\Delta/2} \left(\tfrac{\sin(N\Delta/2)}{\sin(\Delta/2)}\right)
\end{aligned}$$

Therefore the intensity at a given angle is proportional to[4]

[3] Here we need to use the geometric series and Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, or more specifically, $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

[4] Quick reminder: $\left|e^{i\theta}\right| = 1$.

$$\left| e^{i(N-1)\Delta/2} \left(\frac{\sin(N\Delta/2)}{\sin(\Delta/2)} \right) \right|^2 = \frac{\sin^2\left(N\frac{\pi d}{\lambda}\sin(\theta)\right)}{\sin^2\left(\frac{\pi d}{\lambda}\sin(\theta)\right)}$$

The widths of the spikes of this "approximately sinc function squared" are inversely proportional to N. There's one spike for every value θ_k such that $\frac{\pi d}{\lambda}\sin(\theta_k) = k\pi$, which quickly reduces to $\sin(\theta_k) = \frac{k\lambda}{d}$. This means that diffraction gratings put the interference fringes in the same place, but make them sharper.

My diffraction grating claims[5] to have "1000 lines per mm". That means that $d = 10^{-6}m$, $\lambda = 5.32 \times 10^{-7}m$, and $N =$ "a lot". The kth fringe appears when

$$\sin(\theta_k) = \frac{k\lambda}{d} = k\frac{5.32 \times 10^{-7}m}{10^{-6}m} = 0.532k$$

When the wavelength is on the same scale of the slit separation, we can no longer use the small angle approximation. The first fringe appears at

$$\theta_1 = \arcsin(0.532) \approx 0.56 \approx 32^o$$

and since $|\sin(\theta)| \leq 1$, the second fringe doesn't appear at all. We see three (very sharp), widely-spaced interference fringes.

Probability Amplitude

The photo-electric effect is a good way to build single-photon sensors. Photomultipliers apply a "staircase" of voltages between a series of plates so that when a single photon ejects a single electron from the first plate, that electron will be pulled into the second plate with enough kinetic energy to release a dozen electrons, when accelerate toward the third plate, etc.

With an array of single-photon detectors (or more frugally, a single movable detector) we can finally answer a very important question:

"What happens when you do the double slit experiment, one photon at a time?"

The terrifying result is the same interference pattern, built up one point at a time, in proportion to the distribution of energy deposited by the original indistinguishably-many-photons experiment. In other words *individual* photons interfere with *themselves*. This singular "quantum" thing, "the photon", only impacts the screen at a single point and yet it is perfectly described by a phenomena that involves going through both slits at the same time.

[5] Unlike the $0.25mm$ spacing of the double slit, this is a little hard to confirm by eye.

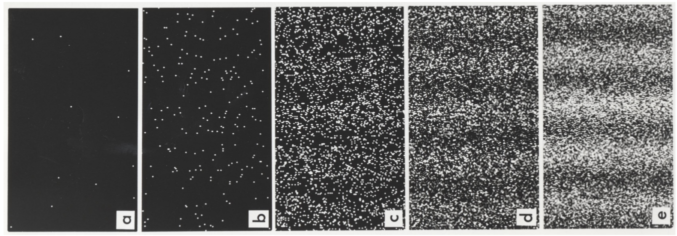

Figure 3: Individual photon impacts, aggregated over time.

This is strange, because we started out describing *waves* of light interfering, and ended up describing the *probability* of a particle of light impacting the screen in a given location.

For waves we can talk about intensity and amplitude, where the amplitude ψ of the wave is what we use to calculate interference and $|\psi|^2$ is the intensity, the actual "amount of light", of the fringes. But for individual particles we have to talk about probabilities and probability amplitudes.

We *infer* the existence of the "**quantum wave function**", ψ, which is a "**probability amplitude**", through measurements of the "probability", P. ψ is generally assumed to have a complex value, and

$$\boxed{P = |\psi|^2}$$

is always a non-negative real number.

In the case of the double slit experiment, we literally set up a field of detectors, count up how often photons impact the screen at each location, and build a probability distribution. It is through direct measurements of probabilities that we *infer* the value of the probability amplitude.

Figure 4: Space is what rulers measure, time is what clocks measure, and probability is what counters measure. As abstract as these ideas may become, it's important to remind yourself what they represent *physically*.

Two Path Interferometer

The essential characteristic that quantum information is based on, is not the "wave-ness" of the double slit experiment, it's the "multiple-state-ness". A single photon literally goes through both slits.[6]

To boil down the important aspect of the double slit experiment, we'll consider the "two path interferometer". Two paths are provided along which light can propagate. Denote the state of being on the bottom path as $|0\rangle$ and on the top as $|1\rangle$. These are called "state vectors" or "**kets**",[7] and they are used to represent quantum states.

Like the double slit experiment, we find that individual photons can be on both paths at the same time, which we call a "**superposition**" of states. The photon isn't simply on the top path or the bottom path; it's in a "linear combination" of both paths. The notation we use for this situation is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are the complex-valued probability amplitudes for the top and bottom paths. In other words, $P(top\,path) = |\alpha|^2$ and $P(bottom\,path) = |\beta|^2$. Because the total probability always sums to one,

$$|\alpha|^2 + |\beta|^2 = 1$$

We call $|\psi\rangle$ a "normalized" state when the sum of the magnitude squared of it's coefficients is one. Unless otherwise specified, assume that any state vector you see is normalized.

This is our first example of a "**qubit**" or "quantum bit", which is a superposition of exactly two distinguishable quantum states, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

In this example, photons enter along the bottom path, encounter a beam splitter, and are sent to one of two detectors. We find, by turning down the intensity low enough to prevent multiple photons, that the detectors never fire at the same time.[8]

This beam splitter evenly divides the incoming photons onto both paths according to the "**Hadamard** operator",[9] H.

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad\qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

[6] Or many slits for diffraction gratings!

[7] The name comes from Dirac's cleverly named "bra-ket" notation. We'll meet the bras soon.

[8] This is a bit ideal. Actual equipment always has some noise or failure rate, but today the best fidelity for the most basic quantum operations on individual particles is north of 99%.

[9] We're playing fast and loose with the complex phase, because 1) the global phase can't be measured (only *relative* phase is important) and 2) we can change the phase at will by changing the lengths of the branches in the diagrams.

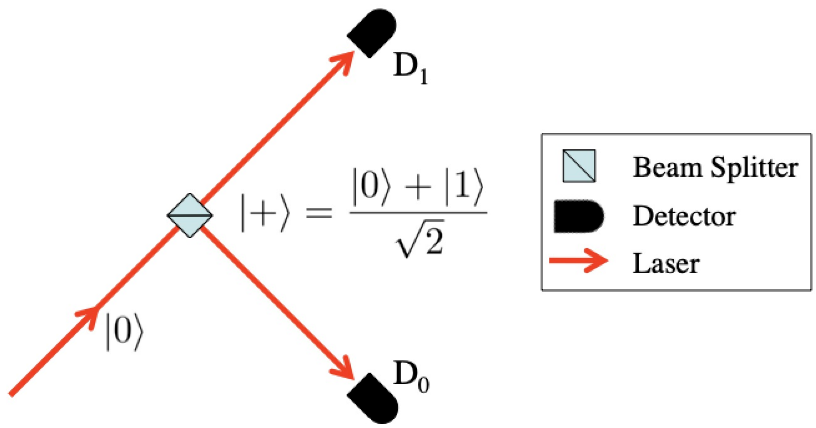

Figure 5: A photon starts in the lower path, is split into a superposition of top and bottom paths, and sent to detectors. Either one or the other clicks for each photon with probability $\frac{1}{2}$.

We can write H more succinctly as a matrix in the $\{|0\rangle, |1\rangle\}$ basis:

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

These two new states will show up many times throughout the course and are important enough that they get their own names

$$\boxed{\begin{array}{cc} |+\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}} & |-\rangle = \dfrac{|0\rangle - |1\rangle}{\sqrt{2}} \\ |0\rangle = \dfrac{|+\rangle + |-\rangle}{\sqrt{2}} & |1\rangle = \dfrac{|+\rangle - |-\rangle}{\sqrt{2}} \end{array}}$$

Notice that the probability of being on either path is $\left|\pm\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$, so this is the probability for either D_0 or D_1 to "click".[10] Notice that this is true for both $|+\rangle$ and $|-\rangle$, so the probability of the two detectors clicking is the same whether the photon is initially $|0\rangle$ or $|1\rangle$. What's the difference between these two states?

One of the bedrock rules of quantum mechanics is that physical operations on quantum states are "linear". An operator, F, is linear if, for any x and y and any constant c:

$$F(x+y) = F(x) + F(y) \qquad\qquad F(cx) = cF(x)$$

[10] "Click" is a term of art, since modern photodetectors don't click when they detect a photon.

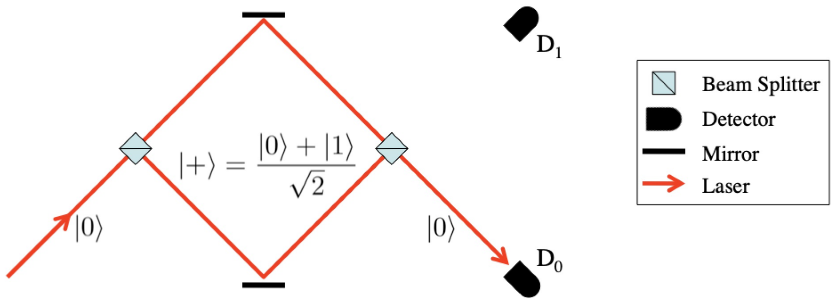

Figure 6: In this set up, the outcome is non-random. Not only can we infer that the photon takes both paths, but we know the particular combination: $|+\rangle$, not $|-\rangle$.

We find that we need that linearity if we put in a second beam splitter and that we get a definite result

$$\begin{aligned}
& H^2|0\rangle \\
= & H|+\rangle \\
= & H\left[\tfrac{|0\rangle+|1\rangle}{\sqrt{2}}\right] \\
= & \tfrac{1}{\sqrt{2}}H|0\rangle + \tfrac{1}{\sqrt{2}}H|1\rangle \qquad \text{(Linearity)} \\
= & \tfrac{1}{\sqrt{2}}\left[\tfrac{|0\rangle+|1\rangle}{\sqrt{2}}\right] + \tfrac{1}{\sqrt{2}}\left[\tfrac{|0\rangle-|1\rangle}{\sqrt{2}}\right] \\
= & \tfrac{1}{2}|0\rangle + \tfrac{1}{2}|1\rangle + \tfrac{1}{2}|0\rangle - \tfrac{1}{2}|1\rangle \\
= & |0\rangle
\end{aligned}$$

and similarly, $H^2|1\rangle = H|-\rangle = |1\rangle$. So the Hadamard transform just switches between the two pairs of states

$$H|0\rangle = |+\rangle \qquad H|+\rangle = |0\rangle$$

$$H|1\rangle = |-\rangle \qquad H|-\rangle = |1\rangle$$

This sort of simple interference implies not only the physical existence of the superposition states $|+\rangle$ and $|-\rangle$, but also demonstrates that they are different.

Why Complex Numbers?

After a 50/50 beam splitter we find that the probability of finding the photon on either path is 0.5. Importantly, even as we change the length of the path, the probability of a detector on that path clicking stays the same.

But changing the length of the path definitely has an effect. As we see in the double slit experiment, the difference in length between two paths affects how they interfere. Starting with the set up described above, if we add a "trombone" section to arbitrarily change the path length, we find that the result smoothly varies back and forth between 100% D_0 clicks and 100% D_1 clicks.

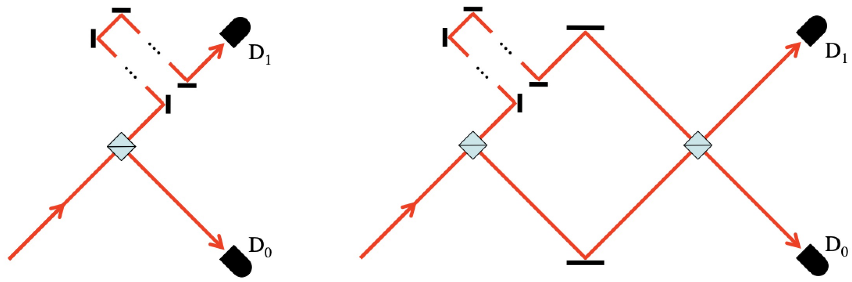

Figure 7: Left: Changing the length of one of the paths has no impact on the probability of finding a photon on that path. Right: Changing the length of one of the paths does have an impact on how the two interfere, swinging the relative contribution of that path smoothly from "+1" to "-1".

In other words, by the time the two paths are recombined, the amplitude of the contribution from the upper path switches sign relative to the bottom path. And yet, because the probability of detecting a photon on the upper path never changes, the amplitude must get between +1 and -1 without changing its magnitude. With real numbers, that's not possible.

This is a thumbnail sketch for why real numbers are insufficient for describing even rudimentary quantum phenomena. Complex numbers evidently do a decent job, but there's no guarantee that some dynamic in the future will demand more. Indeed, quaternions (which are to complex numbers as complex numbers are to real numbers) are useful for describing intrinsic spin. That said, we'll stick to complex-valued amplitudes in this course.

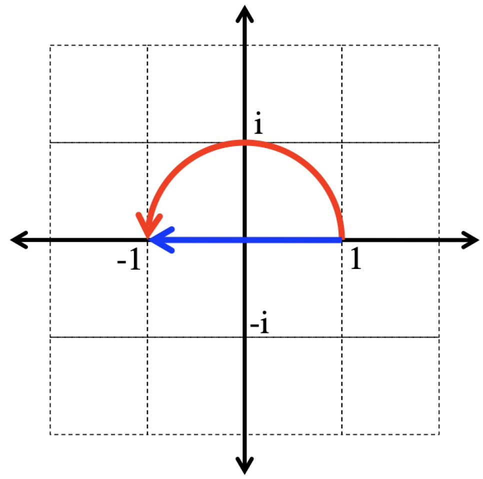

Figure 8: With real numbers (blue), going between ±1 requires you to change magnitude. With complex numbers (red) that's no problem.

Additional (Optional) Reading

There's nothing special about photons. To the best of our ability to measure, literally *everything* exhibits interference patterns when tested by the double slit experiment. This was shown to be true of electrons in 1927 by Davisson-Germer (who scattered electrons off of crystals, which behave like a diffraction grating) and later for Buckyballs (carbon-60 molecules).

In the last few decades the record for largest molecule has been broken repeatedly, the most recent being in 2019 using molecules with up to 2,000 atoms. These aren't the largest objects to demonstrate quantum superposition, just the largest to do it using slits.

"Quantum superposition of molecules beyond 25 kDa" by Fien, Geyer, Zwick, Kiałka, Pedalino, Mayor, Gerlich, and Arndt.

https://doi.org/10.1038/s41567-019-0663-9

Postulate 1

Associated to any isolated physical system is a complex Hilbert space known as the "state space" of the system. The system is described by its "state vector", $|\psi\rangle$, which is a unit vector in the system's state space.

$$|\psi\rangle = \sum_k \alpha_k |k\rangle$$

Exercises

1) Running Interference

The De-Broglie wavelength of a massive object with momentum p is

$$\lambda = \frac{h}{p} = \frac{6.6 \times 10^{-34} \frac{kg\, m^2}{s}}{p}$$

Imagine a never-ending stream of identical coherent people (never mind the details), all with a mass of 100kg and sprinting at 10m/s through a pair of doors placed 10m apart. After passing the two doors, this crowd of people continue running until the reach a finish line.

a) What is the angle between the "fringes" produced by the runners?

b) At the finish line, how physically far apart should these fringes be in order for someone to be able to tell the difference between them?

c) Given your answer to part b, how far away should the finish line be from the two doors?

d) How feasible is this?

2) Dialing For Interference.

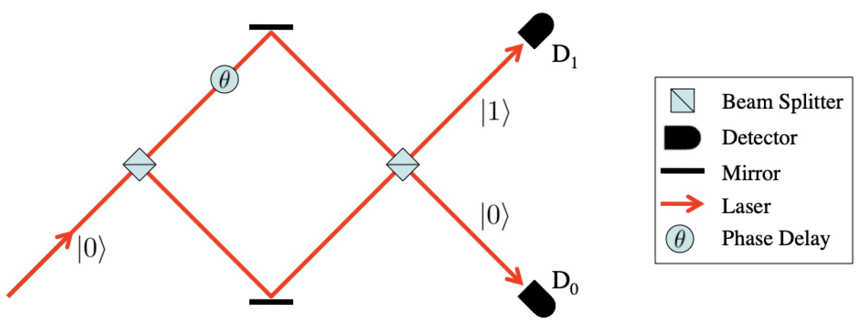

Figure 9: The amplitude of the top path is multiplied by a "complex phase" of $e^{i\theta}$.

By adding a phase delay along the top path,[11] we can control how they interfere. The effect of the phase delay is to multiply the coefficient by $e^{i\theta}$, so in the set up shown the phase delay changes $\alpha|1\rangle \rightarrow \alpha e^{i\theta}|1\rangle$.

[11] Either path works. Only the phase difference between the paths is important.

a) What is the state just before the second beam splitter?

b) What is the state after the second beam splitter?

c) For what values of θ do we find that D_0 always clicks? That D_1 always clicks? That the detectors click equally often?

3) A Very Hair-Trigger Bomb

This is a classic thought experiment (that is also an actual experiment, when there aren't explosives involved). A bomb, so sensitive that it will explode if exposed to <u>any</u> light, may have been placed into the upper path of the interferometer. So we need to see if it's there or not without actually bouncing any light off of it.

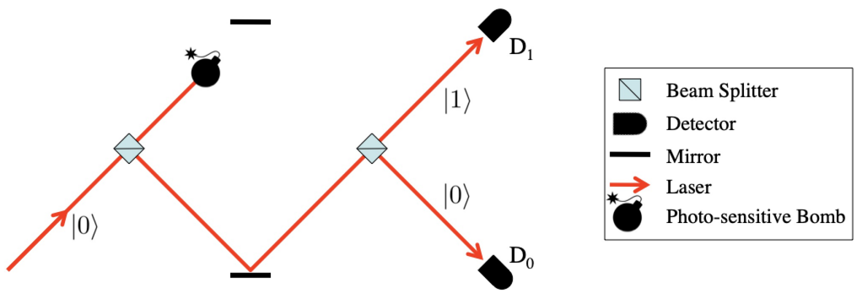

Figure 10: If even a single photon takes the top path, the bomb detonates. In order to detect it we can use interference effects (or the lack thereof). This is called "ghost imaging" or an "interaction-free measurement".

a) If the bomb is <u>not</u> there, what is the probability of each detector clicking?

b) If the bomb is there, what is the probability of the bomb exploding? What are the probabilities of each detector clicking?

c) If D_1 clicks, what can you say about the bomb and what can you say about the path of the photon you detected?