QI Lecture 2

Quantum Mechanics Review
Bra-Ket Notation

Bounded Linear Functions

If f is a bounded linear function that maps quantum states to complex numbers, then we find that it must take a very specific form. Being linear means that $f(\alpha|\eta\rangle + \beta|\phi\rangle) = \alpha f(|\eta\rangle) + \beta f(|\phi\rangle)$, so for any given $|\psi\rangle = \sum_k \alpha_k |k\rangle$

$$f(|\psi\rangle) = f\left(\sum_k \alpha_k |k\rangle\right) = \sum_k \alpha_k f(|k\rangle)$$

In other words, this function is defined entirely by what value it assigns to each state in the basis, $\{|1\rangle, |2\rangle, \ldots\}$. This is rapidly beginning to look like the "dot product".

Being bounded just means that these values, $f(|k\rangle)$, must be finite (and square summable for infinite bases).

Bra-Ket notation

A "bra" is a bounded linear function on the space of quantum states. It takes a ket as input and produces a complex number as output. The space of quantum states is a "**Hilbert space**", meaning that for every ket, $|\psi\rangle$, there is a corresponding bra, $\langle\psi|$. We call $\langle\psi|$ the "**adjoint**" of $|\psi\rangle$, and indicate this relationship using "dagger"

$$[|\psi\rangle]^\dagger = \langle\psi|$$

the dagger indicates the "conjugate transpose". For any complex number, $z = a + bi$, the complex conjugate is $z^* = a - bi$. This is a useful operation in large part because it gives us a cute way of calculating the magnitude of complex numbers, $|z| = \sqrt{a^2 + b^2}$.

$$|z|^2 = zz^* = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$$

The transpose swaps the ket and the bra, $|\psi\rangle \leftrightarrow \langle\psi|$,

$$|\psi\rangle = \sum_j \alpha_j |j\rangle \qquad\qquad [|\psi\rangle]^\dagger = \langle\psi| = \sum_k \alpha_k^* \langle k|$$

We can also think of kets and bras as column and row vectors respectively, to get a sense of how to work with them.

$$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix} \qquad\qquad [|\psi\rangle]^\dagger = \langle\psi| = \begin{pmatrix} \alpha_1^*, & \alpha_2^*, & \cdots \end{pmatrix}$$

Hilbert Spaces have an "inner product", $\langle\phi|\psi\rangle$, which defines the relationship between basis states, $\{|j\rangle\}$, and their unique corresponding "dual basis", $\{\langle j|\}$,

$$\boxed{\langle k|j\rangle = \delta_{kj} = \begin{cases} 1 & ,k = j \\ 0 & ,k \neq j \end{cases}}$$

So for any $|\psi\rangle = \sum_j \alpha_j |j\rangle$ and $|\phi\rangle = \sum_j \beta_j |j\rangle$,

$$\langle\phi|\psi\rangle = \left(\sum_k \beta_k^* \langle k|\right)\left(\sum_j \alpha_j |j\rangle\right) = \sum_{jk} \beta_k^* \alpha_j \langle k|j\rangle = \sum_{jk} \beta_k^* \alpha_j \delta_{jk} = \sum_k \beta_k^* \alpha_k$$

The inner product[1]

$$\boxed{\langle\phi|\psi\rangle = \sum_k \beta_k^* \alpha_k}$$

has a few important properties. First, the Cauchy-Schwarz[2] inequality:

$$|\langle\phi|\psi\rangle|^2 = \left|\sum_m \beta_m^* \alpha_m\right|^2 \leq \left(\sum_j |\beta_j|^2\right)\left(\sum_k |\alpha_k|^2\right) = \langle\phi|\phi\rangle\langle\psi|\psi\rangle$$

For a normalized state, $|\psi\rangle = \sum_k \alpha_k |k\rangle$,

$$\boxed{\langle\psi|\psi\rangle = \sum_k |\alpha_k|^2 = 1}$$

[1] You may recognize this "inner product" from vector calculus or linear algebra as the "dot product". The dot product only exists for finite vector spaces, while the inner product is a more general idea that can be applied in many contexts. For example, an inner product of two functions is $\langle g, f\rangle = \int_a^b g^*(x) f(x)\, dx$.

[2] The fact that quantum states are in L^2, "the space of square integrable functions", or ℓ^2, "the space of square summable sequences", is at the heart of why they form a "Hilbert space". The Cauchy-Schwarz inequality can be generalized to the Hölder inequality, $|\sum_m \beta_m^* \alpha_m| \leq \left(\sum_j |\beta_j|^p\right)^{\frac{1}{p}} \left(\sum_k |\alpha_k|^q\right)^{\frac{1}{q}}$ where $\frac{1}{p} + \frac{1}{q} = 1$. The "dual space" is where the "bras" live, and it's the set of functions or sequences that, when combined as in Hölder or Cauchy with "kets", are always finite. In other words, the dual space of L^p is L^q. Only when $p = q = 2$ is the dual space the same as the original space and that's when you can define an inner product between two objects in the same space (every ket gives you a bra). So it's not a given that the space of quantum states is a Hilbert space; it's an important and special property of square integrable/summable things.

and for pairs of normalized states

$$\boxed{|\langle\phi|\psi\rangle| \leq 1}$$

with equality only for $|\phi\rangle = |\psi\rangle$.

The order of the inner product matters, but not a lot.

$$\langle\phi|\psi\rangle^* = \left(\sum_k \beta_k^* \alpha_k\right)^* = \sum_k \beta_k \alpha_k^* = \langle\psi|\phi\rangle$$

and therefore

$$\boxed{\langle\phi|\psi\rangle^* = \langle\psi|\phi\rangle}$$

We can write and prove this in a slightly different way using the dagger[3] notation:

$$[\langle\phi|\psi\rangle]^\dagger = [|\psi\rangle]^\dagger [\langle\phi|]^\dagger = \langle\psi|\phi\rangle$$

When $\langle\phi|\psi\rangle = 0$, we say that $|\phi\rangle$ and $|\psi\rangle$ are "**orthogonal**", sometimes written "$|\phi\rangle \perp |\psi\rangle$".

The inner product has an important physical significance.

$$\langle k|\psi\rangle = \langle k|\left(\sum_j \alpha_j |j\rangle\right) = \sum_j \alpha_j \langle k|j\rangle = \sum_j \alpha_j \delta_{kj} = \alpha_k$$

In other words, $\langle k|\psi\rangle$ is the probability amplitude of the state $|k\rangle$ in $|\psi\rangle$. If we have a reliable way of detecting the state $|k\rangle$, then $|\langle k|\psi\rangle|^2$ is the probability of seeing $|k\rangle$ when observing $|\psi\rangle$. We can generalize this to any pair of states; the probability of detecting $|\phi\rangle$ given the state $|\psi\rangle$ is

$$\boxed{P = |\langle\phi|\psi\rangle|^2}$$

Example Given the state $|\psi\rangle = \frac{1}{\sqrt{3}}|1\rangle - \frac{1}{\sqrt{3}}|2\rangle + \frac{i}{\sqrt{3}}|3\rangle$, what is the probability of detecting $|\phi\rangle = \frac{1+i}{2}|2\rangle - \frac{1}{\sqrt{2}}|3\rangle$ and $|\eta\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle$?

[3] A quick reminder, the transpose reverses the order of things: $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$.

$$
\begin{aligned}
& P(\phi) \\
= \;& |\langle\phi|\psi\rangle|^2 \\
= \;& \left|0 \cdot \tfrac{1}{\sqrt{3}} + \tfrac{1-i}{2} \cdot \tfrac{-1}{\sqrt{3}} + \tfrac{-1}{\sqrt{2}} \cdot \tfrac{i}{\sqrt{3}}\right|^2 \\
= \;& \left|\tfrac{-1+i}{2\sqrt{3}} + \tfrac{-i}{\sqrt{6}}\right|^2 \\
= \;& \tfrac{1}{12}\left|-1 + i - \sqrt{2}i\right|^2 \\
= \;& \tfrac{1}{12}\left[(-1)^2 + (1-\sqrt{2})^2\right] \\
= \;& \tfrac{1+1-2\sqrt{2}+2}{12} \\
= \;& \tfrac{2-\sqrt{2}}{6} \\
\approx \;& 0.098
\end{aligned}
$$

$$
\begin{aligned}
& P(\eta) \\
= \;& |\langle\eta|\psi\rangle|^2 \\
= \;& \left|\tfrac{1}{\sqrt{2}} \cdot \tfrac{1}{\sqrt{3}} + \tfrac{1}{\sqrt{2}} \cdot \tfrac{-1}{\sqrt{3}} + 0 \cdot \tfrac{i}{\sqrt{3}}\right|^2 \\
= \;& \left|\tfrac{1}{\sqrt{6}} - \tfrac{1}{\sqrt{6}} + 0\right|^2 \\
= \;& |0|^2 \\
= \;& 0
\end{aligned}
$$

In analogy to the inner product on real vector spaces, where it is proportional to cosine of the angle between its input vectors, we say that $|\eta\rangle$ and $|\psi\rangle$ are “orthogonal”.

■

Projections

The introduction of bras immediately begs an interesting question: if $\langle k|k\rangle = 1$, then what is $|k\rangle\langle k|$? The bra on the right is just waiting for a ket to come along,[4] so we may as well give it one.

Give that $|\psi\rangle = \sum_j \alpha_j |j\rangle$,

$$|k\rangle\langle k|\psi\rangle = |k\rangle\alpha_k = \alpha_k|k\rangle$$

This is the kth component of $|\psi\rangle$. We've “projected” $|\psi\rangle$ onto $|k\rangle$.

[4] It's “waiting for a ket to come along”, because acting on kets is the defining property of bras and the only thing they do.

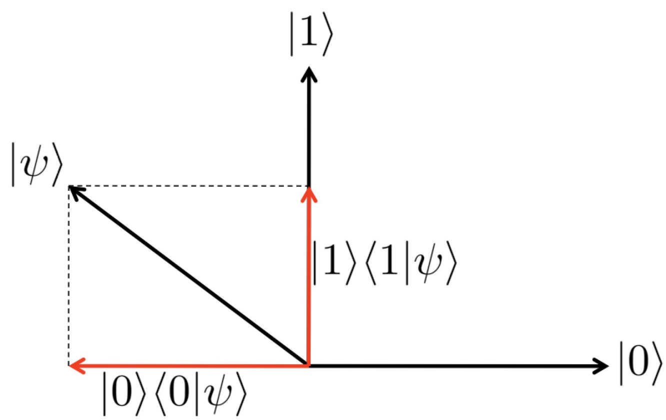

Figure 1: We can picture a projection onto $|k\rangle$ as the "shadow" that $|\psi\rangle$ casts on $|k\rangle$.

Notice that after the first projection, repeated projections do nothing. In other words,

$$(|k\rangle\langle k|)^2 = |k\rangle\langle k|k\rangle\langle k| = |k\rangle\langle k|$$

We can express projects as a matrix. For example, if the basis of the state space is $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$, then

$$|2\rangle\langle 2| = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0, & 1, & 0, & 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For a given basis, we can use projections to write a given state, $|\psi\rangle$, as a combination of any set of basis states. This is easiest to see with an example.

Example We'd like to write the state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

as a combination of $\{|0\rangle, |1\rangle\}$ as well as $\{|+\rangle, |-\rangle\}$. Clearly, we know exactly how it's written in the former basis, but just for completeness:

$$|0\rangle\langle 0|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle\langle 0|0\rangle + \frac{1}{2}|0\rangle\langle 0|1\rangle = \frac{\sqrt{3}}{2}|0\rangle$$

$$|1\rangle\langle 1|\psi\rangle = \frac{\sqrt{3}}{2}|1\rangle\langle 1|0\rangle + \frac{1}{2}|1\rangle\langle 1|1\rangle = \frac{1}{2}|1\rangle$$

In the $\{|+\rangle, |-\rangle\}$ basis

$$|+\rangle\langle+|\psi\rangle = \frac{\sqrt{3}}{2}|+\rangle\langle+|0\rangle + \frac{1}{2}|+\rangle\langle+|1\rangle = \frac{\sqrt{3}}{2\sqrt{2}}|+\rangle + \frac{1}{2\sqrt{2}}|+\rangle = \frac{\sqrt{3}+1}{2\sqrt{2}}|+\rangle$$

$$|-\rangle\langle-|\psi\rangle = \frac{\sqrt{3}}{2}|-\rangle\langle-|0\rangle + \frac{1}{2}|-\rangle\langle-|1\rangle = \frac{\sqrt{3}}{2\sqrt{2}}|-\rangle - \frac{1}{2\sqrt{2}}|-\rangle = \frac{\sqrt{3}-1}{2\sqrt{2}}|-\rangle$$

In other words, the same state $|\psi\rangle$ can be written in both bases like this:

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle = \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)|+\rangle + \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)|-\rangle$$

Based on the diagram (fig. 2), this looks about right.

$|\psi\rangle$ is closer to $|0\rangle$ than $|1\rangle$, and the amplitudes and probabilities reflect that: $P(0) = \left|\frac{\sqrt{3}}{2}\right|^2 = 0.75$ and $P(1) = \left|\frac{1}{2}\right|^2 = 0.25$.

$|\psi\rangle$ is much closer to $|+\rangle$ than $|-\rangle$, and the amplitudes and probabilities reflect that as well: $P(+) = \left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2 \approx 0.933$ and $P(-) = \left|\frac{\sqrt{3}-1}{2\sqrt{2}}\right|^2 \approx 0.067$.

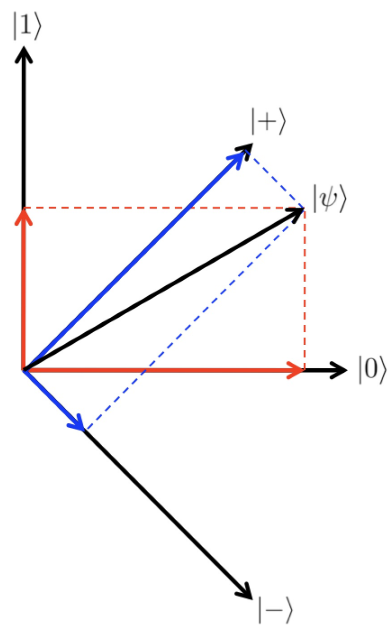

Figure 2: The same state, $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$ in this case, can be written as linear combinations of different bases, $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$ in this case. Using projections we can determine how much of each basis state we need to construct $|\psi\rangle$.

Polarizers (and the subjectivity of superposition)

The polarization of light is a useful toy system to work with, both in theory and practice, and polarizers provide a very clean example of a projection operation.

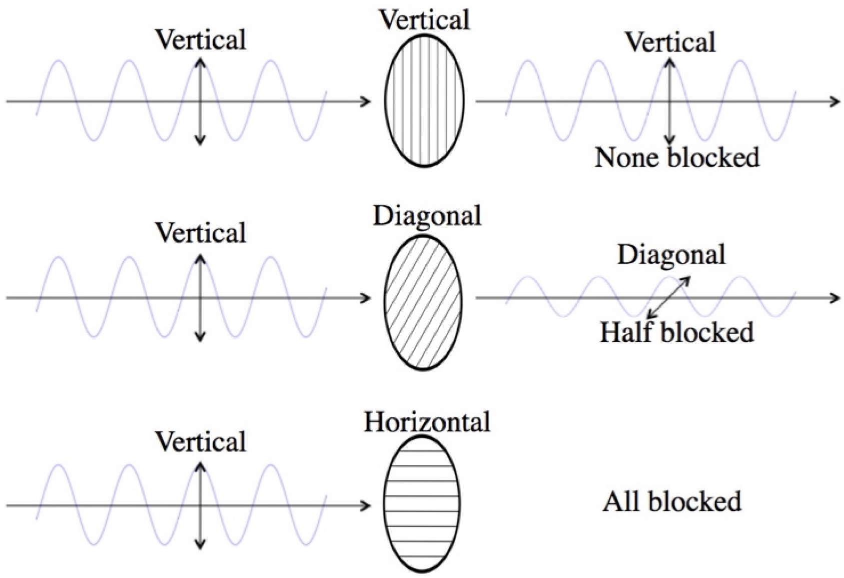

Figure 3: The basic rules for polarized light and polarizers. Top: If the polarizations line up, all of the light gets through. Bottom: If the polarizations are perpendicular, then none of the light gets through. Middle: At other angles there's a chance of light getting through, but what gets through is always oriented in the direction of the polarizer.

The polarization of light is a direction perpendicular to the direction of propagation. We typically use the vertical and horizontal polarization states, $\{|\uparrow\rangle, |\rightarrow\rangle\}$, as the orthonormal basis states for the polarization state.

$|\downarrow\rangle$ is also a vertical polarization state, but we take it to mean $|\downarrow\rangle = -|\uparrow\rangle$. In other words, there is a phase difference[5] of π between $|\downarrow\rangle$ and $|\uparrow\rangle$, because delaying or advancing a transverse wave by a phase of π means switching the direction of the wave (crests become troughs and vice versa).

Generally speaking, we'll identify $|\uparrow\rangle = |0\rangle$ and $|\rightarrow\rangle = |1\rangle$ so that the orthogonality of the polarization states will look more familiar:

$$\boxed{\langle\rightarrow|\uparrow\rangle = \langle 0|1\rangle = 0}$$

[5] We'll be using Euler's equation, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, a lot. Because $e^{i\pi} = -1$, when we say "the phase difference is π", we mean that $|\downarrow\rangle = e^{i\pi}|\uparrow\rangle = -|\uparrow\rangle$.

If the angle of polarization is θ from the vertical, then define the (normalized) polarization state[6]

$$\boxed{|\theta\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle}$$

A vertical polarizer allows vertical photons, $|0\rangle$, to pass but destroys/absorbs horizontal photons, $|1\rangle$. For light polarized at an angle θ to the vertical, the polarizer permits $\cos^2(\theta)$ of the light to get through.

We can describe the effect of a polarizer aligned at angle θ, P_θ, very simply using this projection (or "outer product"),

$$P_\theta = |\theta\rangle\langle\theta|$$

So if we have a beam of vertically polarized light, the effects of a vertical, diagonal, or horizontal polarizer are:

$$P_0|0\rangle = |0\rangle\langle 0|0\rangle = |0\rangle$$

No effect.

$$P_{\frac{\pi}{4}}|0\rangle = |\frac{\pi}{4}\rangle\langle\frac{\pi}{4}|0\rangle = |\frac{\pi}{4}\rangle\left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right)|0\rangle = \frac{1}{\sqrt{2}}|\frac{\pi}{4}\rangle$$

$\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$ chance of passing through, and when it does it is diagonally polarized.

$$P_{\frac{\pi}{2}}|0\rangle = |1\rangle\langle 1|0\rangle = 0$$

Completely blocked.

With complete generality, if the light is polarized at an angle of θ and the polarizer is aligned at an angle of ϕ, then

$$P_\phi|\theta\rangle = \cos(\theta - \phi)|\phi\rangle$$

It's worth stopping to point out that we've just violated a very important rule: these states are <u>not normalized</u>. What we mean (and never do this again) by $\frac{1}{\sqrt{2}}|\frac{\pi}{4}\rangle$ is the state of the light that survives the polarizer and we're ignoring the light that gets destroyed. This is a convenient (and lazy and absolutely wrong) way to describe the state of the light passing through the polarizer, because if we want to look at passing through multiple

[6] Clearly there's some ambiguity in the notation since $|\frac{\pi}{2}\rangle = \cos\left(\frac{\pi}{2}\right)|0\rangle + \sin\left(\frac{\pi}{2}\right)|1\rangle = |1\rangle$. However, since we're very unlikely to have any need for an angle of 1 radian, we'll use $|1\rangle$ to refer to the horizontal polarization state.

polarizers, we don't want to keep track of the probability of the light being absorbed by each individual polarizer or what happens afterward. "If it's not included, it's gone" is fine for our purposes here.

There are two ways to look at diagonally polarized light, $|\frac{\pi}{4}\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. We can look at it as a state in its own right, or as a superposition of vertical and horizontal states. This is yet another example of superposition. Diagonal light is a superposition of vertical and horizontal states just like light taking the two paths in the interferometer or going through the two slits in the double slit experiment are examples of superposition. If the state in a two-path interferometer is $|+\rangle$, then when we measure the path we find that there's a $\frac{1}{2}$ chance of finding it on either path. Similarly, if we have a diagonally polarized photon and measure whether it is vertical or horizontal, we find that it has a $\frac{1}{2}$ chance of being either.

The reason we're looking at this example is that it helps us to shake ourselves free of the idea of a preferred basis; we naturally assume that the states involving multiple paths are superpositions while states involving only one path are not superpositions. But with light polarizations, we find that whether something is in a superposition has as much to do with the state (the angle of polarization) as it does with how we measure the state (the angle of the polarizer).

Being in a superposition is about both the state and the method of measurement.

The Identity

The projection doesn't need to be one-dimensional, $|k\rangle\langle k|$. For example, if $|\psi\rangle = \sum_j \alpha_j|j\rangle$ and $P = |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|$, then

$$P|\psi\rangle = |1\rangle\langle 1|\psi\rangle + |2\rangle\langle 2|\psi\rangle + |3\rangle\langle 3|\psi\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle + \alpha_3|3\rangle$$

So we can project onto any subspace, just by doing a sum of projections onto its basis states. In the example a moment ago, we projected onto the subspace spanned by the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis.

But what happens if you project onto the entire space and not just a subspace? We'll denote this operator

$$\boxed{I = \sum_k |k\rangle\langle k|}$$

Notice what I does.

$$I|\psi\rangle = \left(\sum_k |k\rangle\langle k|\right)\left(\sum_j \alpha_j|j\rangle\right) = \sum_{jk} \alpha_j|k\rangle\langle k|j\rangle = \sum_{jk} \delta_{jk}\alpha_j|k\rangle = \sum_j \alpha_j|j\rangle = |\psi\rangle$$

I is called the "**identity operator**" because it does nothing, which is a lot more useful than it sounds. Most of the operations you're likely to be familiar with have an identity. For example, 1 is the "multiplicative identity" (because $1x = x$) and 0 is the "additive identity" (because $0 + x = x$).

The identity is technically a projection operator, but it projects onto the entire space.

Operators

An operator takes a ket as input and produces a new ket as output, $M|\psi\rangle = |\phi\rangle$. There are a lot of ways to write operators, but we'll try to stick to bra-ket notation as often as possible. In this lecture we'll be going back and forth between bra-ket notation and matrix notation to try to build up intuition about how things work.

When we talk about writing things "in the $\{|\square\rangle, |\cdot\rangle, |\circ\rangle, |\triangle\rangle\}$ basis"[7] we're defining what our vectors mean:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \sim a|\square\rangle + b|\cdot\rangle + c|\circ\rangle + d|\triangle\rangle$$

In general, we can write any operator as a matrix in the $\{|0\rangle, |1\rangle\}$ basis this way:

$$a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1| \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

I'm using "$\sim$" here to mean "equivalent notations", because while

$$\left(\frac{1}{\sqrt{2}}|0\rangle\langle 0| + \frac{1}{\sqrt{2}}|1\rangle\langle 0| + \frac{1}{\sqrt{2}}|0\rangle\langle 1| - \frac{1}{\sqrt{2}}|1\rangle\langle 1|\right)|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

and

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

mean *exactly the same thing* in the $\{|0\rangle, |1\rangle\}$ basis, it doesn't make sense to mix the notation like this[8]

[7] This isn't a typo, I'm just trying to underscore that it doesn't matter what symbols you use to represent your states.

[8] Physicists actually do this all the time, but they shouldn't. It's like using grammar not good. So when you come across this sort of thing, you need to be clear in your own mind about what exactly is being said.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}|1\rangle$$

Exercises

1) Make it up

a) Make up two normalized states that are orthogonal.

b) Make up two normalized states that are unequal and non-orthogonal.

2) What's wrong with this?

$$|\eta\rangle = -\frac{1}{2}i|1\rangle - \frac{2}{3}|2\rangle + \frac{1}{2}|3\rangle \qquad |\nu\rangle = \frac{3}{4}i|1\rangle + \frac{1}{2}|2\rangle - \frac{2}{3}|3\rangle$$

a) Find the inner product.

b) Why is this result strange and what's causing it?

#3) Projection

$$|\psi\rangle = \frac{1-i}{2\sqrt{2}}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$$

a) Given $|\psi\rangle$, what is the probability of measuring $|+\rangle$?

b) Given $|+\rangle$, what is the probability of measuring $|\psi\rangle$?

c) Project $|\psi\rangle$ onto $|+\rangle$.

#4) Carefully Doing Nothing

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$$

a) Calculate $[|0\rangle\langle 0| + |1\rangle\langle 1|]\,|\psi\rangle$ directly.
This is about how the identity changes nothing.

b) Calculate $[|+\rangle\langle +| + |-\rangle\langle -|]\,|\psi\rangle$ directly.
This is also the identity, but your state will now be written in the $\{|+\rangle, |-\rangle\}$ basis.

c) Rewrite the result from part b in the $\{|0\rangle, |1\rangle\}$ basis using the definitions of $|\pm\rangle$.
This is about the fact that the results from a and b are really the same.

5) Quantum Xeno Effect

Assume a beam of vertically polarized light with intensity 1 (units aren't important) is incident on a series of N polarizers, where the jth polarizer is aligned at an angle of $\theta_j = \frac{\pi}{2N} j$ to the vertical, so that the first polarizer is aligned almost vertically and the last is aligned horizontally.

a) What is the polarization of the light exiting the last polarizer?

b) What is the intensity of the light exiting the last polarizer?

c) What is the answer to both questions as $N \to \infty$?