## QI Lecture 3

## Operators and Eigenstates

## Eigenvalues and Eigenstates

One of the most important aspects of an operator are its "eigenvalues" and "eigenstates". Suppose $M$ is an operator, $\lambda$ is a scalar, ${ }^{1}$ and $|v\rangle$ is a state. If

$$
M|v\rangle=\lambda|v\rangle
$$

then $|v\rangle$ is an "eigenstate" of $M$ and $\lambda$ is the corresponding "eigenvalue". Eigenstates/values are incredibly useful, in part because they reduce the action of an operator or matrix to simple multiplication.

Example The vertical-polarization projection operator, $P_{0}=|0\rangle\langle 0|$, has two eigenstates

$$
P_{0}|0\rangle=|0\rangle \quad P_{0}|1\rangle=0|1\rangle
$$

or writing the same thing using matrices in the $\{|0\rangle,|1\rangle\}$ basis

$$
\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]\binom{1}{0}=\binom{1}{0} \quad\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]\binom{0}{1}=\binom{0}{0}
$$

Therefore, the eigenvalues and their corresponding eigenstates are $\{\lambda=1,|0\rangle\}$ and $\{\lambda=0,|1\rangle\}$.

Notice that we can write $P_{0}$ like this:

$$
P_{0}=|0\rangle\langle 0|+0|1\rangle\langle 1|
$$

This is the "spectral decomposition of $P_{0}$ ".

[^0]Example The Pauli operator $X$ (more on that below) acts as a NOT gate: $X|0\rangle=|1\rangle$ and $X|1\rangle=|0\rangle$. We can write this using bras and kets or a matrix in the $\{|0\rangle,|1\rangle\}$ basis:

$$
X=|1\rangle\langle 0|+|0\rangle\langle 1| \sim\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]
$$

$X$ has two eigenvalues, $\lambda= \pm 1$, and two corresponding eigenstates.
The eigenstate for $\lambda=1$ is $|+\rangle$

$$
X|+\rangle=X\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)=\frac{X|0\rangle+X|1\rangle}{\sqrt{2}}=\frac{|1\rangle+|0\rangle}{\sqrt{2}}=|+\rangle
$$

and the eigenstate for $\lambda=-1$ is $|-\rangle$

$$
X|-\rangle=X\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)=\frac{X|0\rangle-X|1\rangle}{\sqrt{2}}=\frac{|1\rangle-|0\rangle}{\sqrt{2}}=-|-\rangle
$$

Notice that we can write the spectral decomposition of $X$ like this:

$$
X=|+\rangle\langle+|-|-\rangle\langle-|
$$

Example The Identity operator, $I$, does nothing; $I|\psi\rangle=|\psi\rangle, \forall|\psi\rangle$. Therefore, $I$ has one eigenvalue, $\lambda=1$, and every state is an eigenstate.

Regardless of the basis, $I$ is written:

$$
I=\left[\begin{array}{llll}
1 & 0 & 0 & \\
0 & 1 & 0 & \\
0 & 0 & 1 & \\
& & & \ddots
\end{array}\right]
$$

That means that we can write $I$ using any basis in the same way. For example, $\{|0\rangle,|1\rangle\}$ and $\{|+\rangle,|-\rangle\}$ are different bases for the same space and

$$
I=|0\rangle\langle 0|+|1\rangle\langle 1|=|+\rangle\langle+|+|-\rangle\langle-|
$$

## Pauli Matrices

The Pauli operator's several representations, their spectral decomposition (more on that in a minute), their matrix representation in the $\{|0\rangle,|1\rangle\}$ basis, and finally their eigenvalues and eigenstates are:

$$
\begin{aligned}
& X=\sigma_{x}=\sigma_{1}=|+\rangle\langle+|-|-\rangle\langle-| \sim\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \begin{cases}\lambda=1 & |+\rangle=\frac{|0\rangle+|1\rangle}{\sqrt{2}} \\
\lambda=-1 & |-\rangle=\frac{|0\rangle-|1\rangle}{\sqrt{2}}\end{cases} \\
& Y=\sigma_{y}=\sigma_{2}=|\circlearrowleft\rangle\langle\circlearrowleft|-|\circlearrowright\rangle\langle\circlearrowright| \sim\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right] \begin{cases}\lambda=1 & |\circlearrowleft\rangle=\frac{|0\rangle+i|1\rangle}{\sqrt{2}} \\
\lambda=-1 & |\circlearrowright\rangle=\frac{|0\rangle-i|1\rangle}{\sqrt{2}}\end{cases} \\
& Z=\sigma_{z}=\sigma_{3}=|0\rangle\langle 0|-|1\rangle\langle 1| \sim\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \begin{cases}\lambda=1 & |0\rangle \\
\lambda=-1 & |1\rangle\end{cases}
\end{aligned}
$$

I'm using $|\circlearrowleft\rangle$ and $|\circlearrowright\rangle$ as the notation for the eigenstates of $Y$ because they can be used to describe "circular polarization" states in the same way that $|0\rangle$ and $|1\rangle$ can be used to describe horizontal and vertical polarization.

## Eigenbases and the Spectral Decomposition

All, or almost all, of the operators that we'll talk about in this class are either "unitary" or "hermitian". Both of these kinds of operators are "normal", meaning that $M^{\dagger} M=M M^{\dagger}$, and are therefore subject to the Spectral Theorem.

Theorem (Spectral Theorem). If the operator $M$ is normal, then the eigenstates of $M$, $\left|v_{j}\right\rangle$, with eignvalues $\lambda_{j}$ can be arranged into an orthonormal basis, $\left\{\left|v_{1}\right\rangle,\left|v_{2}\right\rangle, \ldots\right\}$, and $M$ has a "spectral decomposition" meaning that it can be written

$$
M=\sum_{j} \lambda_{j}\left|v_{j}\right\rangle\left\langle v_{j}\right|
$$

The only ambiguity in this theorem (and the reason for the phrase "can be arranged") comes from repeated or "degenerate" eigenvalues which give rise to "degenerate eigenspaces". Every eigenvalue has an associated eigenstate, but when an eigenvalue, $\lambda$, is repeated and there are two (or more) associated eigenstates, $|v\rangle$ and $|w\rangle$, then every linear combination of those eigenstates is also an eigenstate because

$$
M(\alpha|v\rangle+\beta|w\rangle)=\alpha M|v\rangle+\beta M|w\rangle=\alpha \lambda|v\rangle+\beta \lambda|w\rangle=\lambda(\alpha|v\rangle+\beta|w\rangle)
$$

Two randomly selected states in this $\lambda$-eigenspace are as likely to be orthogonal as two randomly drawn lines on a table. It's easy to chose states that are orthogonal, but you do have to make that choice. For normal operators, eigenstates with distinct eigenvalues are automatically orthogonal.

We'll frequently talk about the spectral decomposition or eigenbasis of operators, but keep in mind that with repeated eigenvalues comes a degenerate eigenspace, so the basis will not be unique, because there's a subtle arbitrary choice being made in the background.

So we really live in a very privileged universe, because quantum states form a Hilbert space and quantum operators always have a spectral decomposition. ${ }^{2}$ These two facts are a best-case scenario for doing math.

Example Notice that as long as the space in question has more than one dimension, the 1 -eigenspace of $I$ is degenerate and therefore when we choose the eigenbasis we can make it non-orthogonal or orthogonal at will.

For example, $\left\{\frac{\sqrt{3}|0\rangle-i|1\rangle}{2}, \frac{|0\rangle+i \sqrt{3}|1\rangle}{2}\right\}^{3}$ is an orthonormal eigenbasis for $I=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$. It's "ortho-" because

$$
\left(\frac{\sqrt{3}\langle 0|+i\langle 1|}{2}\right)\left(\frac{|0\rangle+i \sqrt{3}|1\rangle}{2}\right)=\frac{1}{4}(\sqrt{3}\langle 0 \mid 0\rangle+i 3\langle 0 \mid 1\rangle+i\langle 1 \mid 0\rangle-\sqrt{3}\langle 1 \mid 1\rangle)=\frac{1}{4}(\sqrt{3}-\sqrt{3})=0
$$

and it's "-normal" because

$$
\begin{aligned}
& \left(\frac{\sqrt{3}\langle 0|+i\langle 1|}{2}\right)\left(\frac{\sqrt{3}|0\rangle-i|1\rangle}{2}\right)=\frac{1}{4}(3\langle 0 \mid 0\rangle-i \sqrt{3}\langle 0 \mid 1\rangle+i \sqrt{3}\langle 1 \mid 0\rangle+\langle 1 \mid 1\rangle)=\frac{1}{4}(3+1)=1 \\
& \left(\frac{\langle 0|-i \sqrt{3}\langle 1|}{2}\right)\left(\frac{|0\rangle+i \sqrt{3}|1\rangle}{2}\right)=\frac{1}{4}(\langle 0 \mid 0\rangle+i \sqrt{3}\langle 0 \mid 1\rangle-i \sqrt{3}\langle 1 \mid 0\rangle+3\langle 1 \mid 1\rangle)=\frac{1}{4}(1+3)=1
\end{aligned}
$$

On the other hand, $\left\{|1\rangle, \frac{|0\rangle+i \sqrt{3}|1\rangle}{2}\right\}^{4}$ is a non-orthogonal eigenbasis because

$$
\langle 1|\left(\frac{\sqrt{3}|0\rangle-i|1\rangle}{2}\right)=\frac{\sqrt{3}\langle 1 \mid 0\rangle-i\langle 1 \mid 1\rangle}{2}=-\frac{i}{2} \neq 0
$$

while both states in the basis are eigenstates of $I$ (because every state is an eigenstate of $I$ ).

[^1]
## Unitary Operators

All quantum interactions, including the passage of time, are described using "unitary" operations. ${ }^{5}$ This is a very profound restriction. An operator, $U$, is unitary if and only if

$$
U^{\dagger} U=U U^{\dagger}=I
$$

Although we wouldn't normally do this, we can write the inner product as

$$
[|\phi\rangle]^{\dagger}[|\psi\rangle]=\langle\phi \mid \psi\rangle
$$

and by doing this we can see what makes unitary operations so special. If we apply the same unitary operation to both the bra and ket before taking an inner product we find that their inner product stays the same.

$$
[U|\phi\rangle]^{\dagger}[U|\psi\rangle]=\langle\phi| U^{\dagger} U|\psi\rangle=\langle\phi| I|\psi\rangle=\langle\phi \mid \psi\rangle
$$




Figure 1: Unitary operations, which includes reflections and rotations, leave the inner product invariant. This means that the length of vectors and the angles between them stay the same.

Because $U^{\dagger} U=U U^{\dagger}=I$, unitary operators are normal and subject to the spectral theorem. If $|v\rangle$ is a normalized eigenstate, then

$$
1=\langle v \mid v\rangle=\langle v| U^{\dagger} U|v\rangle=[U|v\rangle]^{\dagger}[U|v\rangle]=[\lambda|v\rangle]^{\dagger}[\lambda|v\rangle]=\lambda^{*} \lambda\langle v \mid v\rangle=\lambda^{*} \lambda=|\lambda|^{2}
$$

Therefore, the eigenvalues of unitary operators always have modulus 1 .

$$
|\lambda|=1
$$

[^2]Example The Pauli matrices are all unitary. The trickiest of these to see is $Y=$ $\left[\begin{array}{cc}0 & -i \\ i & 0\end{array}\right]$. Its conjugate transpose is:

$$
Y^{\dagger}=\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]^{\dagger}=\left(\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]^{T}\right)^{*}=\left[\begin{array}{cc}
0 & i \\
-i & 0
\end{array}\right]^{*}=\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]
$$

Evidently, $Y^{\dagger}=Y$. But to show that $Y$ is unitary we need to show that $Y^{\dagger}=Y^{-1}$ :

$$
Y^{\dagger} Y=\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]=\left[\begin{array}{cc}
-i^{2} & 0 \\
0 & -i^{2}
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]
$$

$Y^{\dagger} Y=I$ and therefore $Y$ is unitary.

## Observables

Observables are actual, physical measurements that you can make on a system. Any particular "observable" is an operator, $M$, and the possible values of the observation are the eigenvalues. Observables are "projective measurements" ${ }^{6}$ and are "hermitian", meaning that $M=M^{\dagger}$. This is important because all hermitian operators are normal, since $M M^{\dagger}=M M=M^{\dagger} M$, and therefore are subject to the spectral theorem, which is always good news.


Figure 2: For the electrons in an atom, the eigenstates, $|v\rangle$, of the Hamiltonian operator are the electron orbitals (left) and the corresponding eigenvalues, $\lambda$, are the energies of those states. We observe those eigenvalues with the atomic spectrum (right).

Typically, $M$ is written

[^3]$$
M=\sum_{m} m P_{m}
$$
where $m$ are the eigenvalues (the possible values of the observation) and $P_{m}$ is the projection operator onto the corresponding eigenspace.

Being hermitian means that the eigenvalues of $M$ are real, because if $M|v\rangle=m|v\rangle$, then $\langle v| M|v\rangle=m\langle v \mid v\rangle=m$ and

$$
m^{*}=\langle v| M|v\rangle^{\dagger}=[|v\rangle]^{\dagger} M^{\dagger}[\langle v|]^{\dagger}=\langle v| M^{\dagger}|v\rangle=\langle v| M|v\rangle=m
$$

The complex conjugate flips the sign of the imaginary part, so if $m^{*}=m$, then


We generally choose to write the spectral decomposition of observables as $M=\sum_{m} m P_{m}$ instead of $M=\sum_{k} \lambda_{k}\left|v_{k}\right\rangle\left\langle v_{k}\right|$ because we need to be very careful about degenerate (repeated) eigenvalues, which weren't a problem for unitary operators. Here's why.

The probability of observing the result $m^{7}$ is given by

$$
p(m)=\langle\psi| P_{m}|\psi\rangle
$$

which is the magnitude squared of the component of $|\psi\rangle$ found in the $m$-eigenspace because ${ }^{8}$

$$
\langle\psi| P_{m}|\psi\rangle=\langle\psi| P_{m}^{\dagger} P_{m}|\psi\rangle=\| P_{m}|\psi\rangle \|^{2}
$$

When the eigenvalue $m$ is observed, the state of the system "collapses"

$$
|\psi\rangle \longrightarrow \frac{P_{m}|\psi\rangle}{\sqrt{p(m)}}
$$

which is the re-normalized component of $|\psi\rangle$ found in the $m$-eigenspace. Another way to say this is
"the state after a projective measurement is every part of the original state consistent with the observation."

[^4]Notice that we're not looking at the probability of $|\psi\rangle=\sum_{j} \alpha_{j}|j\rangle$ being found in particular basis states, we're looking at the probability of $|\psi\rangle$ being found in particular eigenspaces, which are not generally spanned by individual kets.

So while it is true (by the Spectral Theorem) that we can write

$$
M=\lambda_{1}\left|v_{1}\right\rangle\left\langle v_{1}\right|+\lambda_{2}\left|v_{2}\right\rangle\left\langle v_{2}\right|+\lambda_{3}\left|v_{3}\right\rangle\left\langle v_{3}\right|+\ldots
$$

we find that if, for example, we know that $\lambda_{1}=\lambda_{2}$, then it's better to write this as

$$
M=\lambda_{1}\left(\left|v_{1}\right\rangle\left\langle v_{1}\right|+\left|v_{2}\right\rangle\left\langle v_{2}\right|\right)+\lambda_{3}\left|v_{3}\right\rangle\left\langle v_{3}\right|+\ldots=\lambda_{1} P_{1}+\lambda_{3} P_{3}+\ldots
$$

and to write the probability of observing $\lambda_{1}$ as $p\left(\lambda_{1}\right)=\langle\psi| P_{1}|\psi\rangle$ rather than $p\left(\lambda_{1}\right)=$ $\langle\psi|\left(\left|v_{1}\right\rangle\left\langle v_{1}\right|+\left|v_{2}\right\rangle\left\langle v_{2}\right|\right)|\psi\rangle$. This reminds us that the eigenspace, not the individual kets, is the important thing.

Projection operators give us a quick trick for calculating their expectation values. The expectation value, or average, is just the sum of the eigenvalues (what you observe) weighted by probability.

$$
\begin{gathered}
E[M]=\sum_{m} m p(m)=\sum_{m} m\langle\psi| P_{m}|\psi\rangle=\langle\psi|\left(\sum_{m} P_{m}\right)|\psi\rangle=\langle\psi| M|\psi\rangle \\
E[M]=\langle\psi| M|\psi\rangle
\end{gathered}
$$

Note that in this form we can be completely ignorant about what the eigenstates and eigenvalues are! There's no need to find them at all!

Example We can write a diagonal polarization observable using the first Pauli operator

$$
X=|+\rangle\langle+|-|-\rangle\langle-|
$$

This observable assigns " 1 " to light in the polarization state $\left|\frac{\pi}{4}\right\rangle$, and " -1 " to light in the state $\left|-\frac{\pi}{4}\right\rangle$. Let's consider an arbitrary polarization, $|\theta\rangle=\cos (\theta)|0\rangle+\sin (\theta)|1\rangle$, and consider its expectation value. Given this state:

$$
\begin{aligned}
P(1) & =|\langle+\mid \theta\rangle|^{2} \\
& \left.=\left|\left(\frac{\langle 0|+\langle 1|}{\sqrt{2}}\right)\right| \theta\right\rangle\left.\right|^{2} \\
& =\frac{1}{2}|\langle 0 \mid \theta\rangle+\langle 1 \mid \theta\rangle|^{2} \\
& =\frac{1}{2}|\cos (\theta)+\sin (\theta)|^{2} \\
& =\frac{\cos ^{2}(\theta)+2 \sin (\theta) \cos (\theta)+\sin ^{2}(\theta)}{2} \\
& =\frac{1+2 \sin (\theta) \cos (\theta)}{2} \\
& =\frac{1+\sin (2 \theta)}{2}
\end{aligned}
$$

and similarly,

$$
P(-1)=|\langle-\mid \theta\rangle|^{2}=\cdots=\frac{1-\sin (2 \theta)}{2}
$$

We can use these probabilities to calculate the expectation value.

$$
E[X]=\sum_{j} \lambda_{j} P\left(\lambda_{j}\right)=(1)\left(\frac{1+\sin (2 \theta)}{2}\right)+(-1)\left(\frac{1-\sin (2 \theta)}{2}\right)=\sin (2 \theta)
$$

That was a fair amount of work to get to $\sin (2 \theta)$. We'll do this calculation again with $E[X]=\langle\theta| X|\theta\rangle$ using the matrix in the basis $\{|0\rangle,|1\rangle\}$. That way we don't have to bother knowing anything about the eigenstates.

$$
\begin{aligned}
& X=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \quad|\theta\rangle=\binom{\cos (\theta)}{\sin (\theta)} \quad\langle\theta|=\left(\begin{array}{cc}
\cos (\theta), & \sin (\theta))
\end{array}\right. \\
& E[X]=\langle\theta| X|\theta\rangle \\
& =(\cos (\theta), \sin (\theta))\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\binom{\cos (\theta)}{\sin (\theta)} \\
& =(\cos (\theta), \sin (\theta))\binom{\sin (\theta)}{\cos (\theta)} \\
& =\cos (\theta) \sin (\theta)+\sin (\theta) \cos (\theta) \\
& =2 \sin (\theta) \cos (\theta) \\
& =\sin (2 \theta)
\end{aligned}
$$

Notice that when $\theta=\frac{\pi}{4}$ we have $E[X]=1$ (meaning that we always see $|+\rangle$ ), when $\theta=-\frac{\pi}{4}$ we have $E[X]=-1$ (meaning that we always see $|-\rangle$ ), and when $\theta=0$ we have
$E[X]=0$ (meaning that we always see $|+\rangle$ and $|-\rangle$ equally often). That all sounds about right, so this math must be doing its job.

Example Let's consider ambient light passing through three polarizers, angled at 0, $\frac{\pi}{4}$, and $\frac{\pi}{2}$.

A vertical polarizer has this spectral decomposition

$$
P_{0}=|0\rangle\langle 0|=1|0\rangle\langle 0|+0|1\rangle\langle 1|
$$

Suppose we shoot a photon in the linear polarization state $|\theta\rangle=\cos (\theta)|0\rangle+\sin (\theta)|1\rangle$ at this vertical polarizer.
$m=1$ corresponds to observing $|\theta\rangle$ and finding it in the state $|0\rangle$. This means the photon passes through the polarizer.

$$
p(1)=\langle\theta \mid 0\rangle\langle 0 \mid \theta\rangle=\cos ^{2}(\theta)
$$

$m=0$ corresponds to observing $|\theta\rangle$ and finding it in the state $|1\rangle$. This means the photon was destroyed/absorbed by the polarizer.

$$
p(0)=\langle\theta \mid 1\rangle\langle 1 \mid \theta\rangle=\sin ^{2}(\theta)
$$

We choose these eigenvalues, $m=0,1$, because it's nice to be able to use the expectation value to describe the probability of a photon surviving the polarizer.

$$
E\left[P_{0}\right]=\langle\theta| P_{0}|\theta\rangle=\langle\theta|[1|0\rangle\langle 0|+0|1\rangle\langle 1|]|\theta\rangle=\cos ^{2}(\theta)
$$

After the first filter there is a probability of $\frac{1}{2}$ that an initially randomly polarized photon will either exist and be vertically polarized or that it will have been destroyed. We can see this by averaging over all polarization angles:

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi} \cos ^{2}(\theta) d \theta=\frac{1}{2 \pi} \int_{0}^{2 \pi} \sin ^{2}(\theta) d \theta=\frac{1}{2}
$$

Consider the diagonal polarizer, $P_{\frac{\pi}{4}}=\left|\frac{\pi}{4}\right\rangle\left\langle\frac{\pi}{4}\right|=1|+\rangle\langle+|+0|-\rangle\langle-|$. The probability that a photon in the state $|0\rangle$ will make it through this polarizer $(m=1)$ and the state it will be in if it does are:

$$
p(1)=\langle 0 \mid+\rangle\langle+\mid 0\rangle=\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}=\frac{1}{2} \quad \frac{|+\rangle\langle+\mid 0\rangle}{\sqrt{p(1)}}=\frac{\frac{1}{\sqrt{2}}|+\rangle}{\frac{1}{\sqrt{2}}}=|+\rangle
$$

This is the step that was glazed over in the last lecture notes; we should have more carefully handled the normalization.

Similarly, the probability of $|+\rangle$ making it through the third polarizer, $P_{1}=0|0\rangle\langle 0|+$ $1|1\rangle\langle 1|$, and the state it will be in if it does are:

$$
p(1)=\langle+\mid 1\rangle\langle 1 \mid+\rangle=\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}=\frac{1}{2} \quad \frac{|1\rangle\langle 1 \mid+\rangle}{\sqrt{p(1)}}=\frac{\frac{1}{\sqrt{2}}|1\rangle}{\frac{1}{\sqrt{2}}}=|1\rangle
$$

The probability of making it through all three polarizers is $\frac{1}{8}=\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$.

Generally speaking, the easiest way to handle measurements is to write your state in the eigenbasis of the observable. For example, $X$ is an observable in the $\{|+\rangle,|-\rangle\}$ basis, because those are the eigenstates of $X$. Similarly, $Z$ is an observation in the $\{|0\rangle,|1\rangle\}$ basis.

In this way the coefficients, $\alpha_{k}$, in front of the basis states serve their function: they tell us the probability of observing the corresponding state.

## Example

$$
|\psi\rangle=\frac{4}{5}|0\rangle-\frac{3}{5}|1\rangle
$$

What are the probabilities and results of measurements using $X$ and $Z$ as observables?
The spectral decompositions of the observables are:

$$
X=|+\rangle\langle+|-|-\rangle\langle-| \quad Z=|0\rangle\langle 0|-|1\rangle\langle 1|
$$

The eigenvalues are almost entirely irrelevant. The eigenstates are what's important. $|\psi\rangle$ is already in the eigenbasis of $Z$, so we'll start there.

$$
\begin{array}{ll}
p(|0\rangle)=\left|\frac{4}{5}\right|^{2}=\frac{16}{25}=0.64 & |\psi\rangle \longrightarrow \frac{|0\rangle\langle 0 \mid \psi\rangle}{\sqrt{\frac{16}{25}}}=\frac{\frac{4}{5}|0\rangle}{\frac{4}{5}}=|0\rangle \\
p(|1\rangle)=\left|\frac{3}{5}\right|^{2}=\frac{9}{25}=0.36 & |\psi\rangle \longrightarrow \frac{|1\rangle\langle 1 \mid \psi\rangle}{\sqrt{\frac{9}{25}}}=\frac{\frac{3}{5}|1\rangle}{\frac{3}{5}}=|1\rangle
\end{array}
$$

Using $X$ as an observable is easiest in its eigenbasis, $\{|+\rangle,|-\rangle\}$. We can quickly convert to this basis using the identity in this basis, $I=|+\rangle\langle+|+|-\rangle\langle-|$.

The probabilities of seeing these states when using the observable $X$ and the effects of those measurements are:

$$
\begin{array}{ll}
p(|+\rangle)=\left|\frac{7}{5 \sqrt{2}}\right|^{2}=\frac{49}{50}=0.98 & |\psi\rangle \longrightarrow \frac{|+\rangle\langle+\mid \psi\rangle}{\sqrt{\frac{49}{50}}}=\frac{\frac{7}{5 \sqrt{2}}|+\rangle}{\frac{7}{5 \sqrt{2}}}=|+\rangle \\
p(|-\rangle)=\left|\frac{1}{5 \sqrt{2}}\right|^{2}=\frac{1}{50}=0.02 & |\psi\rangle \longrightarrow \frac{|-\rangle\langle-\mid \psi\rangle}{\sqrt{\frac{1}{50}}}=\frac{\frac{1}{5 \sqrt{2}}|-\rangle}{\frac{1}{5 \sqrt{2}}}=|-\rangle
\end{array}
$$

## Postulate 2

The evolution of a closed quantum system is described by a unitary transformation. That is, the state of the system at $t_{1},\left|\psi\left(t_{1}\right)\right\rangle$ is the related to the state of the system at time $t_{2}$, $\left|\psi\left(t_{2}\right)\right\rangle$, by a unitary operator $U$ which depends only on the two times.

$$
\left|\psi\left(t_{2}\right)\right\rangle=U\left|\psi\left(t_{1}\right)\right\rangle
$$

When it is feasible to describe a system using a Hamiltonian, $H,{ }^{9}$ the time evolution of the state in a closed system is described by the "Schrödinger equation",

$$
i \hbar \frac{d|\psi\rangle}{d t}=H|\psi\rangle
$$

where $\hbar=1.05 \times 10^{-34} \frac{m^{2} \mathrm{~kg}}{\mathrm{~s}}$ ("h bar") is an experimentally derived quantity.

## Postulate 3

Quantum measurements are described by a collection of "measurement operators", $\left\{M_{m}\right\}$. These are operators acting on the state space of the system being measured. The index $m$ refers to the measurement outcomes that may occur. If the state of the system is $|\psi\rangle$ immediately before the measurement, then the probability that the result $m$ occurs is given by

$$
p(m)=\langle\psi| M_{m}^{\dagger} M_{m}|\psi\rangle
$$

and the state of the system immediately after the measurement is

$$
\frac{M_{m}|\psi\rangle}{\sqrt{p(m)}}=\frac{M_{m}|\psi\rangle}{\sqrt{\langle\psi| M_{m}^{\dagger} M_{m}|\psi\rangle}}
$$

The measurement operators satisfy a "completeness equation",

$$
I=\sum_{m} M_{m}^{\dagger} M_{m}
$$

which is implied by the fact that the sum of probabilities is equal to one

$$
1=\langle\psi \mid \psi\rangle=\langle\psi|\left(\sum_{m} M_{m}^{\dagger} M_{m}\right)|\psi\rangle=\sum_{m}\langle\psi| M_{m}^{\dagger} M_{m}|\psi\rangle=\sum_{m} p(m)
$$

An important special case is the projective measurement

[^5]$$
M=\sum_{m} m P_{m}
$$
where $P_{m}$ is the projection onto the eigenspace of $M$ with eigenvalue $m$.

## Exercises

## 1) The Pauli Matrices

a) Verify that both of the given eigenstates and eigenvalues for all three Pauli matrices are correct, by showing that $\sigma_{i}|v\rangle=\lambda|v\rangle$ for each.
b) Verify that all three Pauli matrices are normal operators, by showing that $\sigma_{i} \sigma_{i}^{\dagger}=\sigma_{i}^{\dagger} \sigma_{i}$ for each using their matrix form.
c) Repeat part b using the Pauli matrices in bra-ket form.
d) Verify that the eigenstates of each Pauli operator are orthogonal.
e) Using first the matrix form and then the bra-ket form, prove that $\sigma_{i}^{2}=I$ for all three Pauli operators.

## 2) 3D Glasses

In order to create the illusion of depth on a flat movie theater screen, different images need to be fed to each eye. The first 3D glasses did this with color filters, red and blue, but those messed with the color of the movie. Later glasses used linearly polarized light, but those had the drawback that if you tilted your head, the two images would start to bleed into each other (at $\frac{\pi}{4}$ both eyes see both images equally).

Modern 3D glasses use circularly polarized light, the two eigenstates of the $Y$ Pauli operator. These glasses still use a linear polarizer, but the light first passes through a "quarter wave plate", QWP, which delays the light wave by a quarter of a wavelength in one direction. This amounts to multiplication by $i$. Assuming that the QWP is aligned to delay the vertical component of the incoming wave:

$$
Q W P=\left[\begin{array}{ll}
i & 0 \\
0 & 1
\end{array}\right]
$$

So modern "circular polarization" lenses are a QWP, which is a unitary operation that changes the incoming light, followed by a linear polarizer, which performs a projective measurement.
a) What do the states $|\circlearrowleft\rangle$ and $|\circlearrowright\rangle$ become after passing through the QWP?
b) At what angle should the linear polarizers be aligned in front of each eye to only permit the initially $|\circlearrowleft\rangle$ or $|\circlearrowright\rangle$ photons to get to that eye?
c) If you had two pairs of 3D glasses, how could you show that the linear polarizers are located on the side of the lens closer to the eye?
d) What is the probability of a linearly polarized state, $|\theta\rangle$, getting through each lens? If you wore the modern "circularly polarized" lenses to an old "linearly polarized" theater, what would the movie look like?

## 3) Measure Twice

Suppose you do a pair of projective measurements, $M$ followed by $K$, where the results of $M$ are $m_{1}, m_{2}, \ldots$ and the results of $K$ are $k_{1}, k_{2}, \ldots$. This pair of measurements is itself a measurement.
a) For a given state $|\psi\rangle$, what is the probability of getting the results $m_{3}$, then $k_{2}$ ?
b) What is the state of the system after those two measurement results?


[^0]:    1 "Scalar" means "number". There's really no telling why we don't just say "number". The use of $\lambda$ for eigenvalues is ubiquitous for reasons that are lost to history.

[^1]:    ${ }^{2}$ Also life exists, and people fall in love, and... I don't know... rainbows.
    ${ }^{3}$ There is nothing special about this basis and no hidden meaning. It is a randomly selected example.
    ${ }^{4}$ Again, nothing special. Randomly selected.

[^2]:    ${ }^{5}$ Clever students will note that the projection operator is not unitary. We'll get back to that.

[^3]:    ${ }^{6}$ For most physics courses, this is the only kind of measurement worth considering. A more nuanced understanding of measurements will be considered later, but a taste of it can be found in postulate 3 at the end of the lecture.

[^4]:    ${ }^{7}$ You'll frequently hear things like "The probability of looking at $|\psi\rangle$ and seeing $|\phi\rangle$ is $p=|\langle\phi \mid \psi\rangle|^{2}$.", but that's only technically true if the observable in question has an eigenspace spanned by $|\phi\rangle$. Otherwise, the probability of "seeing $|\phi\rangle$ " is zero, since it's not one of the results of the observable.
    ${ }^{8}$ Remember that for projection operators, $P^{2}=P$ and $P^{\dagger}=P$.

[^5]:    ${ }^{9}$ The Hamiltonian, $H$, is not the same as the Hadamard operator, $H$.

