QI Lecture 4

The Bloch Sphere and Quantum Circuits

The Bloch Sphere

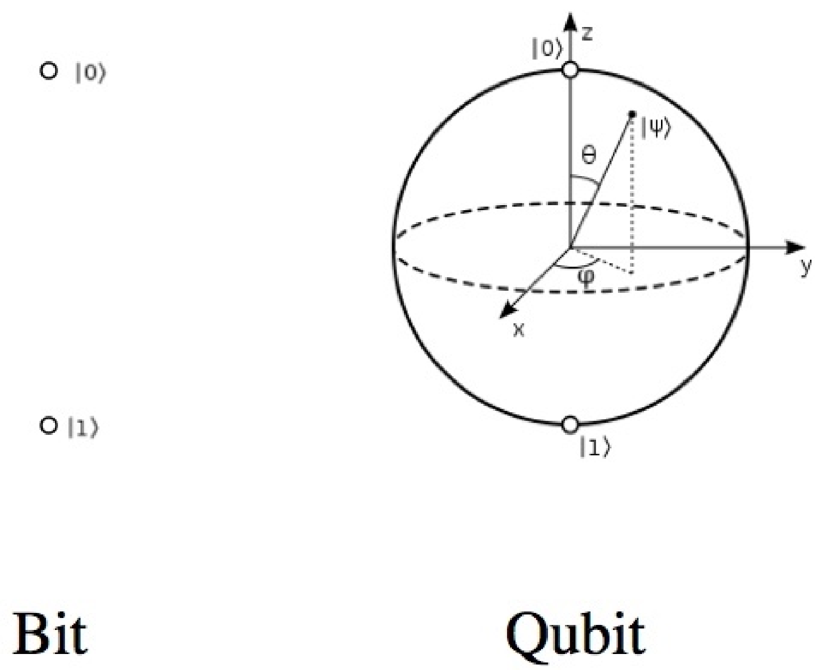

Figure 1: A qubit, $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$, can be represented as a point on the surface of the unit "Bloch Sphere" where θ determines the amplitudes of $|0\rangle$ and $|1\rangle$ and ϕ determines their relative phases.

The Bloch Sphere is a method for visualizing qubits and the effects of single-qubit operations. Because global phase cannot be measured and we're only looking at a single qubit, we can declare that the phase of the $|0\rangle$ is zero (that is, that its amplitude is positive and real).

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Using the Taylor Series of functions, we can define functions of operators. Knowing the Taylor series for e^x, $\cos(x)$, and $\sin(x)$ and using the fact that $X^2 = I$ and $i^2 = -1$, we can quickly calculate

$$\begin{aligned}
R_x(\xi) &\equiv e^{-i\frac{\xi}{2}X} \\
&= \textstyle\sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\frac{\xi}{2}X\right)^n \\
&= I\textstyle\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(-i\frac{\xi}{2}\right)^{2n} + X\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(-i\frac{\xi}{2}\right)^{2n+1} \\
&= I\textstyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(-\frac{\xi}{2}\right)^{2n} + iX\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(-\frac{\xi}{2}\right)^{2n+1} \\
&= \cos\left(-\frac{\xi}{2}\right)I + i\sin\left(-\frac{\xi}{2}\right)X \\
&= \cos\left(\frac{\xi}{2}\right)I - i\sin\left(\frac{\xi}{2}\right)X
\end{aligned}$$

Doing the same to the other Pauli matrices we have the "**rotation operators**", which rotate states in the Bloch sphere around $\hat{x}$, $\hat{y}$, and $\hat{z}$. Each rotation leaves one axis invariant, up to a phase of $e^{-i\frac{\xi}{2}}$. The rotation axis of each operator corresponds to the $\lambda = 1$ eigenstate. That is, since $X|+\rangle = |+\rangle$ it follows[1] that $R_x(\xi)|+\rangle = e^{-i\frac{\xi}{2}X}|+\rangle = e^{-i\frac{\xi}{2}}|+\rangle$ and therefore $\hat{x} = |+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, the rotation axis of $R_x(\xi)$, stays where it is.

The three rotation operators, their matrix representations, and their rotation axes are:

$$\begin{array}{rclcc|c}
R_x(\xi) & = & \cos\left(\frac{\xi}{2}\right)I - i\sin\left(\frac{\xi}{2}\right)X & = & \begin{bmatrix} \cos\left(\frac{\xi}{2}\right) & -i\sin\left(\frac{\xi}{2}\right) \\ -i\sin\left(\frac{\xi}{2}\right) & \cos\left(\frac{\xi}{2}\right) \end{bmatrix} & \hat{x} = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \\
R_y(\xi) & = & \cos\left(\frac{\xi}{2}\right)I - i\sin\left(\frac{\xi}{2}\right)Y & = & \begin{bmatrix} \cos\left(\frac{\xi}{2}\right) & -\sin\left(\frac{\xi}{2}\right) \\ \sin\left(\frac{\xi}{2}\right) & \cos\left(\frac{\xi}{2}\right) \end{bmatrix} & \hat{y} = \frac{|0\rangle+i|1\rangle}{\sqrt{2}} \\
R_z(\xi) & = & \cos\left(\frac{\xi}{2}\right)I - i\sin\left(\frac{\xi}{2}\right)Z & = & \begin{bmatrix} e^{-i\frac{\xi}{2}} & 0 \\ 0 & e^{i\frac{\xi}{2}} \end{bmatrix} & \hat{z} = |0\rangle
\end{array}$$

Example Find a single rotation that moves $|+\rangle$ to $|1\rangle$.

To do this problem take a hard look at figure 1 at each step.

[1] It follows through the Taylor series. If $f(x) = \sum_k a_k x^k$ and $M|v\rangle = \lambda|v\rangle$, then $f(M)|v\rangle = \sum_k a_k M^k|v\rangle = \sum_k a_k \lambda^k|v\rangle = f(\lambda)|v\rangle$.

$|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$ corresponds to $(\theta,\phi) = \left(\frac{\pi}{2},0\right)$, which is the $\hat{x}$ direction. $|1\rangle$ is at the bottom of the sphere, at $(\theta,\phi) = (\pi,*)$. It looks like we should rotate by $\xi = \frac{\pi}{2}$ around $\hat{y}$, but do we want $\xi = \frac{\pi}{2}$ or $\xi = -\frac{\pi}{2}$?

The sign of a rotation is defined by the "**right hand rule**": point the thumb of your right hand along the axis of rotation and your fingers will curl in the positive direction. Doing that in this case, we find that we want a rotation of $\xi = +\frac{\pi}{2}$ in the $\hat{y}$ direction. In other words, $R_y\left(\frac{\pi}{2}\right)$.

Let's try this out and see what happens.

$$R_y\left(\frac{\pi}{2}\right)|+\rangle \sim \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sim |1\rangle$$

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We can create a rotation around an arbitrary unit direction, $\hat{n} = (n_x, n_y, n_z)$, using

$$R_{\hat{n}}(\xi) \equiv e^{-i\frac{\xi}{2}\hat{n}\cdot\vec{\sigma}} = \cos\left(\frac{\xi}{2}\right)I - i\sin\left(\frac{\xi}{2}\right)(n_xX + n_yY + n_zZ)$$

where $\vec{\sigma} \equiv (X,Y,Z)$ is a "vector of Pauli matrices".

Theorem (Z-Y decomposition for a single qubit)**.** *If U is a unitary operation on a single qubit, then there exist real numbers α, β, γ, and δ such that*

$$U = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)$$

Since U is unitary, the rows and columns are orthonormal, from which it follows that there exist real numbers α, β, γ, and δ such that

$$U = \begin{bmatrix} e^{i\left(\alpha-\frac{\beta}{2}-\frac{\delta}{2}\right)}\cos\left(\frac{\gamma}{2}\right) & -e^{i\left(\alpha-\frac{\beta}{2}+\frac{\delta}{2}\right)}\sin\left(\frac{\gamma}{2}\right) \\ e^{i\left(\alpha+\frac{\beta}{2}-\frac{\delta}{2}\right)}\sin\left(\frac{\gamma}{2}\right) & e^{i\left(\alpha+\frac{\beta}{2}+\frac{\delta}{2}\right)}\cos\left(\frac{\gamma}{2}\right) \end{bmatrix}$$

and the fact that this is equal to $e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)$ follows from matrix multiplication.

QED

Multiple Qubits

The hardest part of working with multiple qubits is keeping track of what applies where. At every step be sure to have in your head an idea of which qubit is which and what operation is affecting what.

The state of a single qubit can be written $\alpha|0\rangle + \beta|1\rangle$ and the basis of the space of of a single qubit is $\{|0\rangle, |1\rangle\}$. Describing multiple qubits is more complicated than merely describing them one at a time. There are a couple of ways to combine disparate spaces A and B: the "direct sum" and the "tensor product".

Suppose that

$$A = Span\{|\bigcirc\rangle, |\square\rangle\} \qquad B = Span\{|\blacktriangle\rangle, |\blacklozenge\rangle, |\bigstar\rangle\}$$

This means that if $|\psi\rangle \in A$, then $|\psi\rangle = \alpha|\bigcirc\rangle + \beta|\square\rangle$ for some unique α and β. Same idea for B.

The direct sum, $\oplus$, of two spaces (which you won't have to worry about again) is probably how you imagine combining spaces.

$$A \oplus B = Span\{|\bigcirc\rangle, |\square\rangle, |\blacktriangle\rangle, |\blacklozenge\rangle, |\bigstar\rangle\}$$

This means that if $|\psi\rangle \in A \oplus B$, then $|\psi\rangle = \alpha|\bigcirc\rangle + \beta|\square\rangle + \gamma|\blacktriangle\rangle + \delta|\blacklozenge\rangle + \epsilon|\bigstar\rangle$. In general, if A is m dimensional and B is n dimensional, then their direct sum, $A \oplus B$, is $m + n$ dimensional.[2] So the direct sum of a line and a plane is a volume.

The only reason I'm bothering to mention direct sums is so that you have a better idea of what the tensor product is and isn't. In fact, for the rest of this course "$\oplus$" will mean "addition modulo 2".

The tensor product, $\otimes$, of two spaces is

$$A \otimes B = Span\{|\bigcirc\rangle|\blacktriangle\rangle, |\bigcirc\rangle|\blacklozenge\rangle, |\bigcirc\rangle|\bigstar\rangle, |\square\rangle|\blacktriangle\rangle, |\square\rangle|\blacklozenge\rangle, |\square\rangle|\bigstar\rangle\}$$

In other words, the basis of the product space is a set of new states composed of every pairing of basis states from the two original spaces. To save a little room, these states are often written

$$A \otimes B = Span\{|\bigcirc\blacktriangle\rangle, |\bigcirc\blacklozenge\rangle, |\bigcirc\bigstar\rangle, |\square\blacktriangle\rangle, |\square\blacklozenge\rangle, |\square\bigstar\rangle\}$$

This means that if $|\psi\rangle \in A \otimes B$, then $|\psi\rangle = \alpha|\bigcirc\blacktriangle\rangle + \beta|\bigcirc\blacklozenge\rangle + \gamma|\bigcirc\bigstar\rangle + \delta|\square\blacktriangle\rangle + \epsilon|\square\blacklozenge\rangle + \zeta|\square\bigstar\rangle$. In general, if A is m dimensional and B is n dimensional, then their tensor product, $A \otimes B$, is mn dimensional.[3]

Whenever considering two quantum systems, the combined system is always described using the tensor product of the individual systems. This, it turns out, is at the heart of why "quantum parallelism" is so powerful. More on that later.

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[2] Hence the name "direct <u>sum</u>".

[3] Hence the name "direct <u>product</u>".

It is *extremely* important to keep the order of the kets straight. That allows us to do algebra like this:

$$\frac{|00\rangle + |10\rangle - |01\rangle - |11\rangle}{2} = \frac{|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle - |1\rangle)}{2} = \frac{(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)}{2} = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

You will frequently see subscripts used to help keep track of which qubits are which

$$\frac{|0\rangle_a|0\rangle_b + |1\rangle_a|0\rangle_b - |0\rangle_a|1\rangle_b - |1\rangle_a|1\rangle_b}{2} = \left(\frac{|0\rangle_a + |1\rangle_a}{\sqrt{2}}\right)\left(\frac{|0\rangle_b - |1\rangle_b}{\sqrt{2}}\right)$$

In other words, if the states in systems $A = Span\{|a_1\rangle, \ldots\}$ and $B = Span\{|b_1\rangle, \ldots\}$ are

$$|\psi\rangle = \sum_j \alpha_j |a_j\rangle \qquad |\phi\rangle = \sum_k \beta_k |b_k\rangle$$

then those two systems considered together are their tensor product, $A \otimes B$, and the state of this combined system is:

$$\boxed{|\psi\rangle|\phi\rangle = \left(\sum_j \alpha_j |a_j\rangle\right)\left(\sum_k \beta_k |b_k\rangle\right) = \sum_{j,k} \alpha_j \beta_k |a_j\rangle|b_k\rangle}$$

This truly is the god of physics smiling down on us; you'd think that combining disparate quantum systems would be complicated, but it really isn't.

Ignoring Global Phase

We don't worry about global phase in general, because when we actually measure a quantum state we measure probabilities and $P = |\langle\phi|\psi\rangle|^2 = |e^{i\theta}\langle\phi|\psi\rangle|^2$.

With the math for composite states in hand, we begin to get a sense for why we can be so cavalier about ignoring the global phase in the Bloch Sphere. When you multiply a subspace of a product space by a complex phase, you multiply the entire state by that phase. For example,

$$\left[e^{i\theta}|\psi\rangle\right]|\phi\rangle = \left(\sum_j e^{i\theta}\alpha_j |a_j\rangle\right)\left(\sum_k \beta_k |b_k\rangle\right) = \sum_{j,k} e^{i\theta}\alpha_j \beta_k |a_j\rangle|b_k\rangle = e^{i\theta}\sum_{j,k} \alpha_j \beta_k |a_j\rangle|b_k\rangle = e^{i\theta}\left[|\psi\rangle|\phi\rangle\right]$$

In general, if $\{|j\rangle_k\}_j$ are the basis states for the space A_k, then the basis states for the product space, $A_1 \otimes \ldots \otimes A_n$, are $\{|j_1\rangle_1|j_2\rangle_2 \ldots |j_n\rangle_n\}_{j_1 j_2 \ldots}$

If you multiply all the basis states for just one of those spaces by the same phase, then the basis states for the entire product space get multiplied by the same phase. This is why we can be so casual about the rotations on the Bloch Sphere ignoring global phase.

For example, if we have the state $\frac{|00\rangle+i|01\rangle-|10\rangle+|11\rangle}{2}$ and we multiply the first qubit by $e^{i\frac{\pi}{2}} = i$ we get

$$\frac{i|00\rangle - |01\rangle - i|10\rangle + i|11\rangle}{2}$$

or if we multiply the second qubit by i we get the exact same thing

$$\frac{i|00\rangle - |01\rangle - i|10\rangle + i|11\rangle}{2}$$

and in either case, we can pull that i out front and then ignore it

$$i\left(\frac{|00\rangle + i|01\rangle - |10\rangle + |11\rangle}{2}\right)$$

because global phase doesn't mater.

Quantum Circuits

As far as quantum information theory is concerned, it doesn't matter how a qubit is formed. The same is true in computer science; bits take the form of currents, magnetic domains, charge differentials, pulses of light, etc., but all of these boil down to 0 or 1.

During the first lecture we considered a qubit formed by a photon taking one of two paths, which was subjected to a pair of Hadamard transforms by means of a beam splitter, and finally measured. But regardless of what form the qubit physically takes, how the Hadamard transformation is realized, or how the measurement is done, we draw this simple 1-qubit circuit the same way.

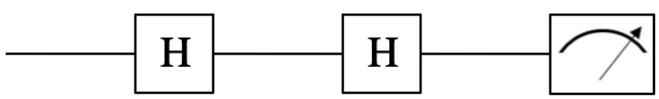

Figure 2: Quantum circuit diagrams are read left-to-right. Here a qubit goes through two Hadamard gates and is then measured.

The simplest two qubit gate is the "CNOT" or "controlled not" or "CX" gate.[4] If the "control" is $|0\rangle$, then the "target" is left alone, and if the control is $|1\rangle$, then the target is flipped.

[4]CX is used because the Pauli X operator switches $|0\rangle \leftrightarrow |1\rangle$, which is NOT, so CX is just another way of saying CNOT.

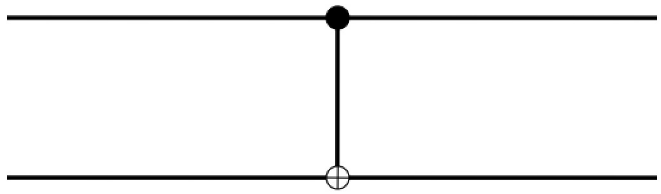

Figure 3: The CNOT gate with the control qubit, indicated with a $\bullet$, affecting the target qubit, indicated with a $\oplus$.

In order to model the affect of the CNOT gate we need to look at the joint space of two qubits, which can be described using the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Assuming that the first qubit is the control and the second is the target

$$CNOT = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$$

or as a matrix

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Example What is the affect of CNOT on a completely arbitrary pair of control and target qubits, $|\psi_c\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\psi_t\rangle = \gamma|0\rangle + \eta|1\rangle$?

First we'll write the state of the two qubits together as a single system.

$$|\psi_c\rangle|\psi_t\rangle = (\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \eta|1\rangle) = \alpha\gamma|00\rangle + \alpha\eta|01\rangle + \beta\gamma|10\rangle + \beta\eta|11\rangle$$

Now we'll apply the CNOT gate.

$$CX|\psi_c\rangle|\psi_t\rangle = \alpha\gamma|00\rangle + \alpha\eta|01\rangle + \beta\gamma|11\rangle + \beta\eta|10\rangle$$

Done.

Notice that before the application of the CNOT (or "CX") gate we could write the state of the system as a whole as the product of two separate states and afterward we could not. Not surprisingly the former is called a "separable state" and the latter is a "non-separable state". More on that later!

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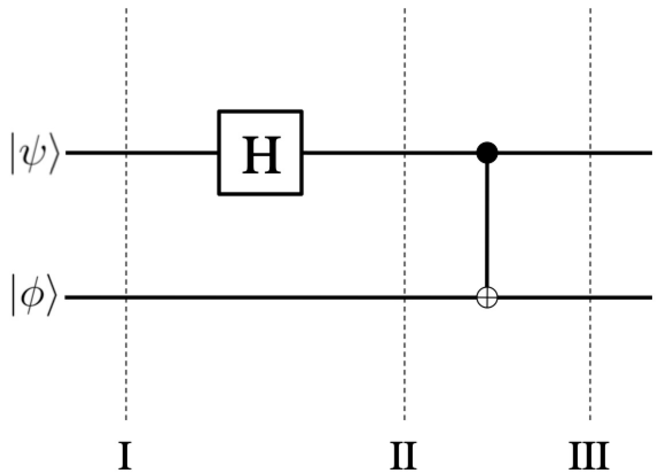

Figure 4: Two qubits are introduced to a circuit on the left and the result is produced on the right. To keep track of the state of the system we calculate the state before and after each operation, at I, II, and III.

Example What is the output of the circuit in figure 4 given that $|\psi\rangle = |\phi\rangle = |1\rangle$? The initial state is

$$I: \qquad |\psi\rangle|\phi\rangle = |1\rangle|1\rangle = |11\rangle$$

The Hadamard gate is applied only to the first qubit, and since $H|1\rangle = |-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$,

$$II: \qquad |-\rangle|1\rangle = \frac{|01\rangle - |11\rangle}{\sqrt{2}}$$

The operation we actually just performed is $H \otimes I$, which is the Hadamard operation applied to the first qubit and nothing (the Identity) applied to the second.

The CNOT gate swaps the second qubit when the first is 1 and leaves it alone when the first is 0.

$$III: \qquad \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

So the output is $\frac{|01\rangle-|10\rangle}{\sqrt{2}}$ which just happens to be $|\Psi^-\rangle$, one of the four "maximally entangled two qubit Bell states". More on that later!

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Postulate 4

The state space of a composite physical system, A, is the tensor product of the state spaces of the individual physical systems, $A_1, \ldots, A_n$.

$$A = A_1 \otimes A_2 \otimes \ldots \otimes A_n$$

Moreover, if system k is prepared in the state $|\psi_k\rangle$, then the joint state of the total system is

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$$

Exercises

1) Three CNOTs.

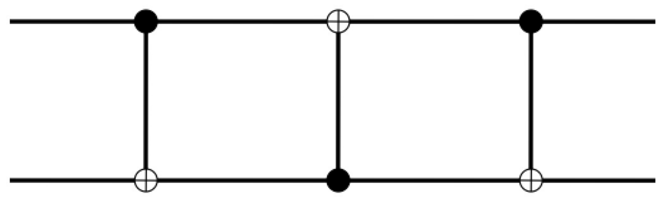

Figure 5: What does this do?

Two arbitrary qubits, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\phi\rangle = \gamma|0\rangle + \eta|1\rangle$, are fed into the inputs of this circuit. By stepping through one gate at a time, figure out what this circuit does. Describe it in words.

2) Composite Space.

$$|\psi\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|0\rangle \qquad\qquad |\phi\rangle = |-\rangle$$

a) $|\psi\rangle \otimes |\phi\rangle = ?$

b) $(H \otimes X)\, |\psi\rangle \otimes |\phi\rangle = ?$

3) Hadamard by Rotation

Find values of α, β, γ, and δ such that you can write the Hadamard operation as $H = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$.

4) Degeneracy of CNOT

a) Write the spectral decomposition of the CNOT operator.

b) CNOT has a degenerate $\lambda = 1$ eigenspace. Use this fact to write the spectral decomposition another way.

5) No Control

If $|a\rangle_c$ is the control qubit and $|b\rangle_t$ is the target qubit, then the effect of a controlled Z gate is:

$$CZ|0\rangle_c|b\rangle_t = |0\rangle_c|b\rangle_t \qquad\qquad CZ|1\rangle_c|b\rangle_t = (I \otimes Z)\,|1\rangle_c|b\rangle_t$$

Show that the CZ gate doesn't actually make a distinction between the control and target qubits.