QI Lecture 8

Entanglement

Separability

We've seen previously that when looking at a pair of quantum systems, A and B, the composite system is the tensor product of the two, $A \otimes B$, and that if $|\psi\rangle_a = \sum_k \alpha_k |k\rangle_a \in A$ and $|\phi\rangle_b = \sum_j \beta_j |j\rangle_b \in B$ (where $\{|k\rangle_a\}$ and $\{|j\rangle_b\}$ are the respective bases) are the states of A and B, then the state of the composite system is

$$|\psi\rangle_a \otimes |\phi\rangle_b = \underbrace{\left(\sum_k \alpha_k |k\rangle_a\right)}_{\in A} \underbrace{\left(\sum_j \beta_j |j\rangle_b\right)}_{\in B} = \underbrace{\sum_{jk} \alpha_k \beta_j |k\rangle_a |j\rangle_b}_{\in A \otimes B}$$

where $\{|k\rangle_a |j\rangle_b\}$ are the basis states of $A \otimes B$.

Notice the form of the above state; we can write it as a tensor product of separate states in A and B. Very reasonably, such a state is called "**separable**".

Separable states have a very important property. Given the full density matrix

$$\rho = |\psi\rangle_a |\phi\rangle_b \langle\psi|_a \langle\phi|_b = \left(\sum_{jk} \alpha_k \beta_j |k\rangle_a |j\rangle_b\right) \left(\sum_{im} \alpha_i^* \beta_m^* \langle i|_a \langle m|_b\right)$$

the reduced density matrix (over A in this case) is

$$\begin{aligned}
\rho_a &= Tr_b[\rho] \\
&= \textstyle\sum_r \langle r|_b \left[\sum_{jk} \alpha_k \beta_j |k\rangle_a |j\rangle_b\right] \left[\sum_{im} \alpha_i^* \beta_m^* \langle i|_a \langle m|_b\right] |r\rangle_b \\
&= \textstyle\sum_r \left[\sum_{jk} \alpha_k \beta_j |k\rangle_a \langle r|j\rangle_b\right] \left[\sum_{im} \alpha_i^* \beta_m^* \langle i|_a \langle m|r\rangle_b\right] \\
&= \textstyle\sum_r \left[\sum_{jk} \alpha_k \beta_j |k\rangle_a \delta_{rj}\right] \left[\sum_{im} \alpha_i^* \beta_m^* \langle i|_a \delta_{mr}\right] \\
&= \textstyle\sum_r \left[\sum_k \alpha_k \beta_r |k\rangle_a\right] \left[\sum_i \alpha_i^* \beta_r^* \langle i|_a\right] \\
&= \textstyle\left[\sum_r |\beta_r|^2\right] \left[\sum_k \alpha_k |k\rangle_a\right] \left[\sum_i \alpha_i^* \langle i|_a\right] \\
&= \textstyle\left[\sum_k \alpha_k |k\rangle_a\right] \left[\sum_i \alpha_i^* \langle i|_a\right] \\
&= |\psi\rangle\langle\psi|
\end{aligned}$$

which means that for separable states, the component of the state existing in one space has no impact on the statistics of the component of the state existing in the other. $|\phi\rangle$ can be anything at all, since $|\psi\rangle\langle\psi|$ (which describes all of the statistics that we can measure in the space A) has no dependence on $|\phi\rangle$ whatsoever.

Separable states are not entangled and non-separable states are entangled.

Bell States

Bell states are the simplest examples of entanglement. They are "maximally entangled bipartite states". A "bipartite state" is a pure state on $A \otimes B$ and an "m-partite state" is a pure state on $A_1 \otimes A_2 \otimes \ldots \otimes A_m$.

$$|B_{00}\rangle = |\Phi_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad |B_{10}\rangle = |\Phi_-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|B_{01}\rangle = |\Psi_+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \qquad |B_{11}\rangle = |\Psi_-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Example If we assume that $|\Phi_+\rangle$ is separable, then we should be able to find α, β, γ, and η such that

$$\frac{|0\rangle_a|0\rangle_b + |1\rangle_a|1\rangle_b}{\sqrt{2}} = [\alpha|0\rangle_a + \beta|1\rangle_a]\,[\gamma|0\rangle_b + \eta|1\rangle_b] = \alpha\gamma|0\rangle_a|0\rangle_b + \alpha\eta|0\rangle_a|1\rangle_b + \beta\gamma|1\rangle_a|0\rangle_b + \beta\eta|1\rangle_a|1\rangle_b$$

$$\Rightarrow \begin{cases} \frac{1}{\sqrt{2}} &= \alpha\gamma \\ 0 &= \alpha\eta \\ 0 &= \beta\gamma \\ \frac{1}{\sqrt{2}} &= \beta\eta \end{cases}$$

Immediately we can see an issue. If $0 = \alpha\eta$, then one or both of these scalars are zero and therefore $\alpha\gamma = 0 \neq \frac{1}{\sqrt{2}}$ and/or $\beta\eta = 0 \neq \frac{1}{\sqrt{2}}$.

This is not a procedure you'd actually want to go through every time you want to check to see if a given state is separable or entangled. We do it here only to verify that non-separable states (the Bell states in particular) exist. In fact, in the completely general case there is no known technique for looking at a state and determining whether or not it is separable.

A state from a product space, chosen at random, is overwhelmingly likely to be non-separable.[1]

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Among the more profound/confusing properties of entangled states is that, although they are a superposition of states whose measurements produce fundamentally random results, measurements of each qubit are correlated.

Example If Alice and Bob each have one qubit of the Bell state $|\Phi_+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and never communicate, what do they see when they each make a measurement in the computational basis?[2]

We did this example in lecture five, when density matrices were introduced. If the density matrix on $A \otimes B$ is $\rho = |\Phi_+\rangle\langle\Phi_+|$, then the reduced density matrix on A is

$$\begin{aligned}
\rho_a &= Tr_b[\rho] \\
&= \left\{ \begin{array}{l} \langle 0|_b \left(\frac{|0\rangle_a|0\rangle_b+|1\rangle_a|1\rangle_b}{\sqrt{2}}\right)\left(\frac{\langle 0|_a\langle 0|_b+\langle 1|_a\langle 1|_b}{\sqrt{2}}\right)|0\rangle_b \\ +\langle 1|_b \left(\frac{|0\rangle_a|0\rangle_b+|1\rangle_a|1\rangle_b}{\sqrt{2}}\right)\left(\frac{\langle 0|_a\langle 0|_b+\langle 1|_a\langle 1|_b}{\sqrt{2}}\right)|1\rangle_b \end{array}\right. \\
&= \left(\frac{|0\rangle_a}{\sqrt{2}}\right)\left(\frac{\langle 0|_a}{\sqrt{2}}\right) + \left(\frac{|1\rangle_a}{\sqrt{2}}\right)\left(\frac{\langle 1|_a}{\sqrt{2}}\right) \\
&= \frac{1}{2}|0\rangle_a\langle 0|_a + \frac{1}{2}|1\rangle_a\langle 1|_a
\end{aligned}$$

Bob sees the same reduced density matrix (just with "b" subscripts), so both see $|0\rangle$ and $|1\rangle$ exactly half the time. Notice that since $|\Phi_+\rangle\langle\Phi_+|$ is a pure state, the source of these $\frac{1}{2}$ probabilities is fundamental randomness, as opposed to classical "I don't know which state it is" randomness.

Since Alice and Bob don't communicate, this is the end of the story. All they can say is that they get a random result.

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Example Alice and Bob each have one qubit of the Bell state $|\Phi_+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and they each make a measurement in the computational basis. What is the probability of each combination of results? What is the effect on the state?

"A measurement in the computational basis" typically means using $Z_a = |0\rangle_a\langle 0|_a - |1\rangle_a\langle 1|_a$ (the Pauli Z matrix) as an observable, since $|0\rangle$ and $|1\rangle$ are eigenstates of Z with

[1]Technically, it's guaranteed in the sense that separable states have "zero measure" in the same way that integers have zero measure on the set of real numbers.

[2]Rather than write "$\{|0\rangle, |1\rangle\}$ basis" over and over, we say "computational basis".

different eigenvalues. To apply this on the product space, $A \otimes B$, we use $Z_a \otimes I_b$. The idea behind using I_b is 1) we don't want to do anything to B, since Alice doesn't have access to it, and 2) if $M|v\rangle = \lambda|v\rangle$, then $(M_a \otimes I_b)|v\rangle_a|\phi\rangle_b = \lambda|v\rangle_a|\phi\rangle_b$ for any $|\phi\rangle_b \in B$, so the eigenstates of the observable are still eigenstates regardless of what's happening in B.

What's the probability that Alice sees $|0\rangle$ and what is the impact on the state? According to Postulate 3,

$$p(|0\rangle_a) = \langle\Phi_+|P_0|\Phi_+\rangle = \left(\frac{\langle 0|_a\langle 0|_b + \langle 1|_a\langle 1|_b}{\sqrt{2}}\right)(|0\rangle_a\langle 0|_a \otimes I_b)\left(\frac{|0\rangle_a|0\rangle_b + |1\rangle_a|1\rangle_b}{\sqrt{2}}\right) = \frac{1}{2}$$

and the effect of Alice's observation of $|0\rangle$ on the state is

$$|\Phi_+\rangle \longrightarrow \frac{P_0|\Phi_+\rangle}{\sqrt{p(|0\rangle_a)}} = \frac{(|0\rangle_a\langle 0|_a \otimes I_b)\left(\frac{|0\rangle_a|0\rangle_b+|1\rangle_a|1\rangle_b}{\sqrt{2}}\right)}{\sqrt{\frac{1}{2}}} = |0\rangle_a\langle 0|0\rangle_a|0\rangle_b + |0\rangle_a\langle 0|1\rangle_a|1\rangle_b = |0\rangle_a|0\rangle_b$$

Similarly, there's a $\frac{1}{2}$ probability that Alice will see $|1\rangle$ leaving the system as a whole in the state $|1\rangle_a|1\rangle_b$.

If Bob now makes a measurement, there is a probability of 1 that his result will be the same as Alice's, because the state is now $|0\rangle_a|0\rangle_b$ or $|1\rangle_a|1\rangle_b$.

A much simpler way to do this problem is to look at the Bell state, $|\Phi_+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$, and simply recognize that the notation is literally telling us "the probability of Alice and Bob seeing the state $|0\rangle_a|0\rangle_b$ is $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$."[3]

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We have to be very careful here, because we've made a subtle but very important assumption: that Alice and Bob are both making <u>objective</u> observations, meaning that whatever they see is now <u>the</u> state as far as the entire universe is concerned. Under this assumption, Alice's observation of $|\Phi_+\rangle$ "collapsed" the state to $|1\rangle_a|1\rangle_b$ meaning that Bob's observation is no longer random!

To handle this more carefully and equitably, we need to treat Alice and Bob as isolated quantum states themselves. We'll look at this later.

Example If Alice and Bob each have one qubit from $|\Phi_+\rangle$ and they both do a measurement in the $\{|\theta\rangle, |\theta\perp\rangle\}$ basis, where

[3] Assuming they're measuring in the computational basis.

$$|\theta\rangle \equiv \cos(\theta)|0\rangle + \sin(\theta)|1\rangle \qquad\qquad |\theta \perp\rangle \equiv -\sin(\theta)|0\rangle + \cos(\theta)|1\rangle$$

then what is the probability that they'll see the same result?

There are a number of ways to do this, but the simplest is to use the method at the end of the last example. We'll write $|\Phi_+\rangle$ in the $\{|\theta\rangle, |\theta \perp\rangle\}$ basis and see what it looks like. To find the components of $|\Phi_+\rangle$ in this new basis, we just project onto the new basis vectors. The bases for Alice and Bob are $\{|\theta\rangle, |\theta \perp\rangle\}$, so the basis for the space $A \otimes B$ is $\{|\theta\rangle_a|\theta\rangle_b, |\theta\rangle_a|\theta \perp\rangle_b, |\theta \perp\rangle_a|\theta\rangle_b, |\theta \perp\rangle_a|\theta \perp\rangle_b\}$ and we want to find the coefficients for

$$|\Phi_+\rangle = \alpha|\theta\rangle_a|\theta\rangle_b + \beta|\theta\rangle_a|\theta \perp\rangle_b + \gamma|\theta \perp\rangle_a|\theta\rangle_b + \eta|\theta \perp\rangle_a|\theta \perp\rangle_b$$

Knowing that $\langle\theta|0\rangle = \cos(\theta)$, $\langle\theta|1\rangle = \sin(\theta)$, $\langle\theta \perp |0\rangle = -\sin(\theta)$, and $\langle\theta \perp |1\rangle = \cos(\theta)$ we can find these coefficients quickly.

$$\alpha = \langle\theta|_a\langle\theta|_b \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) = \frac{\cos^2(\theta) + \sin^2(\theta)}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\beta = \langle\theta|_a\langle\theta \perp |_b \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) = \frac{-\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)}{\sqrt{2}} = 0$$

$$\gamma = \langle\theta \perp |_a\langle\theta|_b \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) = \frac{-\sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta)}{\sqrt{2}} = 0$$

$$\eta = \langle\theta \perp |_a\langle\theta \perp |_b \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) = \frac{\sin^2(\theta) + \cos^2(\theta)}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Therefore

$$|\Phi_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|\theta\rangle_a|\theta\rangle_b + |\theta \perp\rangle_a|\theta \perp\rangle_b}{\sqrt{2}}$$

This is incredibly profound (and will be important for quantum cryptography)! Regardless of how Alice and Bob measure $|\Phi_+\rangle$, as long as they measure in the same basis they'll get the same result!

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Example What if Alice and Bob each have a qubit from $|\Phi_+\rangle$ and Alice measures hers in the $\{|0\rangle,|1\rangle\}$ and Bob measures his in the $\{|\theta\rangle,|\theta\perp\rangle\}$?

Remember:

$$|\theta\rangle \equiv \cos(\theta)|0\rangle + \sin(\theta)|1\rangle \qquad |\theta\perp\rangle \equiv -\sin(\theta)|0\rangle + \cos(\theta)|1\rangle$$

We'll do the same trick we did in the last example, but this time we'll write $|\Phi_+\rangle$ in the $\{|0\rangle_a|\theta\rangle_b, |0\rangle_a|\theta\perp\rangle_b, |1\rangle_a|\theta\rangle_b, |1\rangle_a|\theta\perp\rangle_b\}$ basis:

$$|\Phi_+\rangle = \alpha|0\rangle_a|\theta\rangle_b + \beta|0\rangle_a|\theta\perp\rangle_b + \gamma|1\rangle_a|\theta\rangle_b + \eta|1\rangle_a|\theta\perp\rangle_b$$

Using the same inner product from the last example, we can calculate this quickly.

$$\alpha = \langle 0|_a\langle\theta|_b\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right) = \frac{\cos(\theta)}{\sqrt{2}}$$

$$\beta = \langle 0|_a\langle\theta\perp|_b\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right) = \frac{-\sin(\theta)}{\sqrt{2}}$$

$$\gamma = \langle 1|_a\langle\theta|_b\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right) = \frac{\sin(\theta)}{\sqrt{2}}$$

$$\eta = \langle 1|_a\langle\theta\perp|_b\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right) = \frac{\cos(\theta)}{\sqrt{2}}$$

Therefore

$$|\Phi_+\rangle = \frac{\cos(\theta)|0\rangle_a|\theta\rangle_b - \sin(\theta)|0\rangle_a|\theta\perp\rangle_b + \sin(\theta)|1\rangle_a|\theta\rangle_b + \cos(\theta)|1\rangle_a|\theta\perp\rangle_b}{\sqrt{2}}$$

If Bob identifies $|\theta\rangle$ with "0" and $|\theta\perp\rangle$ with "1", then Alice and Bob will get the same result with probability

$$p(same) = \left|\frac{\cos(\theta)}{\sqrt{2}}\right|^2 + \left|\frac{\cos(\theta)}{\sqrt{2}}\right|^2 = \cos^2(\theta)$$

In the last example we found that it doesn't matter what basis Alice and Bob use, so this is a statement about the likelihood that Alice and Bob will get the same result given that their bases are rotated *with respect to each other* by θ.

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The Bell States form a basis for the space of two qubits. $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ is the "**computational basis**" and since each of these can be easily written as linear combinations of the Bell states (e.g., $|00\rangle = \frac{1}{\sqrt{2}}\left[|\Phi_+\rangle + |\Phi_-\rangle\right]$) we find that

$$\boxed{\{|\Phi_+\rangle, |\Phi_-\rangle, |\Psi_+\rangle, |\Psi_-\rangle\}}$$

is an orthonormal basis of the space of two qubits.

Example If Alice and Bob have one of the four Bell states and they're in the same lab, so they have access to both qubits, how can they figure out which state they have?

"Just measure it!" isn't great advice. Typically, the computational basis is *defined* such that it's easy to measure. For example, we've used vertical and horizontal polarizations for the computational basis many times already and this is a good choice since perpendicular linear polarizations are easy to distinguish.

So this question really boils down to "how do you transform the Bell states into computational basis states, so that they're easy to measure?"

One of the most standard techniques is to use this circuit.

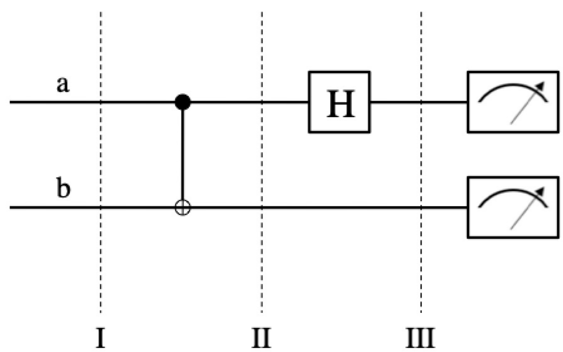

Figure 1: Feeding the two qubits of a Bell state into the "wires" on the left results in a definite measurement in the computational basis, as represented by the "meters" on the right.

Following each Bell state through the circuit we find that the CNOT leave the second qubit in a definite state, meaning we can separate it from the first, and that the first is left in either the $|+\rangle$ or $|-\rangle$ state. Transforming back and forth between $|0\rangle \leftrightarrow |+\rangle$ and $|1\rangle \leftrightarrow |-\rangle$ is exactly what the Hadamard operation does.

$$
\begin{array}{lccccc}
 & I & & II & & III \\
|B_{00}\rangle = & \frac{|00\rangle+|11\rangle}{\sqrt{2}} & \rightarrow & \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|0\rangle & \rightarrow & |00\rangle \\
|B_{01}\rangle = & \frac{|01\rangle+|10\rangle}{\sqrt{2}} & \rightarrow & \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|1\rangle & \rightarrow & |01\rangle \\
|B_{10}\rangle = & \frac{|00\rangle-|11\rangle}{\sqrt{2}} & \rightarrow & \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)|0\rangle & \rightarrow & |10\rangle \\
|B_{11}\rangle = & \frac{|01\rangle-|10\rangle}{\sqrt{2}} & \rightarrow & \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)|1\rangle & \rightarrow & |11\rangle
\end{array}
$$

Suddenly the subscript notation for the Bell states should make a lot more sense.

Notice that unlike all of the previous examples, we're getting a definite result. The reason is that in order to get a definite result you need to be able to measure a state in the "correct basis". For example, if you want to get a definite result while measuring a vertically polarized photon, you can't use a diagonally polarized polarizer. It must be vertical or horizontal to measure in the $\{|\uparrow\rangle, |\rightarrow\rangle\}$ basis.

With Bell states, no measurement on a single qubit is ever in the "correct basis". To get a definite result, you always need both qubits.

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Preview of Von Neumann Entropy

When we write a density matrix as $\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|$, the states $|\psi_k\rangle$ can be anything (other than equal), but in the spectral decomposition of ρ the states need to be orthogonal. Assuming the spectral decomposition is

$$\rho = \sum_k \lambda_k |v_k\rangle\langle v_k|$$

the Von Neumann entropy is

$$S[\rho] = -\sum_k \lambda_k \log(\lambda_k)$$

In the last example we found that when a Bell state is measured in the Bell basis it yields a definite result. The entropy of a Bell state is

$$S[|\Phi_+\rangle\langle\Phi_+|] = -1\log(1) = 0$$

In the second example we found that the reduced density matrix of a Bell state is $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$. The entropy of only one of the two qubits in a Bell state is

$$S\left[\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\right] = -\frac{1}{2}\log\left(\frac{1}{2}\right) - \frac{1}{2}\log\left(\frac{1}{2}\right) = 1$$

So the randomness of part of an entangled state as compared to the randomness (or not randomness) of the whole thing is a good way to measure entanglement.

Exercises

#1) Maybe with a crowbar.

$$|\nu\rangle = \frac{-4\sqrt{3}|00\rangle - i3\sqrt{3}|01\rangle + i4|10\rangle - 3|11\rangle}{10} \qquad |\eta\rangle = \frac{-4\sqrt{3}|00\rangle + i3\sqrt{3}|01\rangle + i4|10\rangle - 3|11\rangle}{10}$$

Which (if either) of these states is separable? Which is entangled?

#2) Unnecessary use of Bells.

a) Given the state $|11\rangle$, what is the probability of seeing each of the Bell states?

b) If you run $|11\rangle$ through the circuit in figure 1, what is the state at III?

#3) Circular Bells.

a) Write $|\Phi_+\rangle$ using circularly polarized states, $|\circlearrowleft\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ and $|\circlearrowright\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$.

b) What does the answer to part a mean?

#4) A modular carillon.[4]

a) Find a superposition of Bell states that is separable.

b) Write $|\psi\rangle = \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}}$ in the Bell basis.

[4] A carillon is a bunch of bells and (presumably) a modular carillon can be separated.