QI Lecture 13

Quantum Fourier Transform

In physics we get to play around with Fourier[1] Transforms at first in the context of position and momentum, where we use it to translate back and forth between the position and momentum wave functions, or indeed when studying waves in general. The Fourier Transform translates a signal into the frequencies of that signal. So for example, if a signal is constant, then its Fourier transform is concentrated at zero.

Figure 1: If $f(t)$ describes the physical back-and-forth movement of sound waves, then the keys of a piano are like the Fourier Transform, $\hat{f}(k)$, describing the frequencies of that sound.

If x_j is a sequence of complex numbers, then X_k, the terms in the sequence of the **Discrete Fourier Transform**, are given by

$$X_k \equiv \sum_{j=0}^{N-1} x_j e^{\frac{2\pi i}{N} jk}$$

Parseval's Theorem says that, if x_j and y_j are sequences with discrete Fourier transforms X_k and Y_k, then

$$\sum_j x_j y_j^* = \frac{1}{N} \sum_k X_k Y_k^*$$

[1] "Fourier" is pronounced "4 E yay", because Monsieur Jean-Baptiste Joseph Fourier was very French.

This looks a lot like a comparison of the inner products before and after the DFT and, except for that $\frac{1}{N}$ term, they're equal. In this class we're interested in the coefficients of states rather than sequences of numbers.

Quantum Fourier Transform

By taking into account that $\frac{1}{N}$ term we can define the **Quantum Fourier Transform** (QFT) as a unitary operation. Rather than write "QFT", we'll use $\mathcal{F}$ to represent the QFT. On each basis state:

$$\boxed{\mathcal{F}|j\rangle \equiv \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} jk} |k\rangle}$$

or equivalently

$$\mathcal{F} \equiv \frac{1}{\sqrt{N}} \sum_{j,k=0}^{N-1} e^{\frac{2\pi i}{N} jk} |k\rangle\langle j|$$

and therefore for any given state $|\psi\rangle = \sum_{j=0}^{N-1} \alpha_j |j\rangle$ the QFT is

$$\boxed{|\hat{\psi}\rangle = \mathcal{F}|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(\sum_{j=0}^{N-1} \alpha_j e^{\frac{2\pi i}{N} jk} \right) |k\rangle}$$

The matrix for the QFT thus has a fairly simple form. If the basis of the state space is $\{|0\rangle, |1\rangle, |2\rangle, \ldots, |N-1\rangle\}$ and $\omega \equiv e^{\frac{2\pi i}{N}}$, then

$$\mathcal{F} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

We may sometimes use $\mathcal{F}_N$ to indicate the dimension of the space, whenever it isn't obvious from context.

Parseval's theorem tells us that for any pair of states,

$$\boxed{\langle\phi|\psi\rangle = \langle\phi|\mathcal{F}^\dagger\mathcal{F}|\psi\rangle = \langle\hat{\phi}|\hat{\psi}\rangle}$$

which is a defining characteristic of unitary operators.

The QFT has an inverse operation called, not surprisingly, the "**Inverse Quantum Fourier Transform**".

$$\boxed{\mathcal{F}^{-1} \equiv \frac{1}{\sqrt{N}} \sum_{j,k=0}^{N-1} e^{-\frac{2\pi i}{N} jk} |k\rangle\langle j|}$$

Notice that the only difference between $\mathcal{F}$ and $\mathcal{F}^{-1}$ is a negative in the exponent. That's because, as a unitary operator,

$$\mathcal{F}^{-1} = \mathcal{F}^{\dagger} = \frac{1}{\sqrt{N}} \sum_{j,k=0}^{N-1} \left(e^{\frac{2\pi i}{N} jk} |k\rangle\langle j| \right)^{\dagger} = \frac{1}{\sqrt{N}} \sum_{j,k=0}^{N-1} e^{-\frac{2\pi i}{N} jk} \left[\langle j| \right]^{\dagger} \left[|k\rangle \right]^{\dagger} = \frac{1}{\sqrt{N}} \sum_{j,k=0}^{N-1} e^{-\frac{2\pi i}{N} jk} |j\rangle\langle k|$$

Switching $j \leftrightarrow k$ makes no difference here, since these are just indices (dummy variables).

From now on we'll assume that $N = 2^n$, because this permits us to use the "fast Fourier transform" (which is easy to implement with a circuit) and because it's a natural result of using n qubits.

We find that it's useful to write

$$|x\rangle = |x_1\rangle|x_2\rangle \dots |x_n\rangle$$

where $x_1 x_2 \dots x_n$ is the binary expansion of x. In other words,

$$x = x_1 2^{n-1} + x_2 2^{n-2} + \dots + x_{n-1} 2 + x_n$$

Example Find the QFT of $|\Phi_-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}|0\rangle + 0|1\rangle + 0|2\rangle - \frac{1}{\sqrt{2}}|3\rangle$.[2]

There are two qubits here, so $N = 2^2 = 4$, and we have that $\alpha_0 = -\alpha_3 = \frac{1}{\sqrt{2}}$ and $\alpha_1 = \alpha_2 = 0$.

[2]Here we're taking advantage of "binary expansion notation", so $|3\rangle \equiv |1\rangle|1\rangle$.

$$
\begin{aligned}
QFT|\Phi_-\rangle &= \tfrac{1}{\sqrt{N}} \textstyle\sum_{k=0}^{N-1} \left(\sum_{j=0}^{N-1} \alpha_j e^{\frac{2\pi i}{N} jk}\right) |k\rangle \\
&= \left\{ \begin{array}{ll} & \frac{1}{2}\left(\sum_{j=0}^{3} \alpha_j\right)|0\rangle \\ + & \frac{1}{2}\left(\sum_{j=0}^{3} \alpha_j e^{i\frac{\pi}{2}j}\right)|1\rangle \\ + & \frac{1}{2}\left(\sum_{j=0}^{3} \alpha_j e^{i\pi j}\right)|2\rangle \\ + & \frac{1}{2}\left(\sum_{j=0}^{3} \alpha_j e^{i\frac{3\pi}{2}j}\right)|3\rangle \end{array} \right. \\
&= \left\{ \begin{array}{ll} & \frac{1}{2}\left(\sum_{j=0}^{3} \alpha_j\right)|0\rangle \\ + & \frac{1}{2}\left(\sum_{j=0}^{3} \alpha_j (i)^j\right)|1\rangle \\ + & \frac{1}{2}\left(\sum_{j=0}^{3} \alpha_j (-1)^j\right)|2\rangle \\ + & \frac{1}{2}\left(\sum_{j=0}^{3} \alpha_j (-i)^j\right)|3\rangle \end{array} \right. \\
&= \left\{ \begin{array}{ll} & \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)|0\rangle \\ + & \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(-i)\right)|1\rangle \\ + & \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(-1)\right)|2\rangle \\ + & \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(i)\right)|3\rangle \end{array} \right. \\
&= \boxed{\tfrac{1+i}{2\sqrt{2}}|1\rangle + \tfrac{1}{\sqrt{2}}|2\rangle + \tfrac{1-i}{2\sqrt{2}}|3\rangle}
\end{aligned}
$$

■

Example Demonstrate that the QFT preserves the normality of $|\psi\rangle = \frac{1}{2}|0\rangle - \frac{i}{2}|1\rangle + \frac{1}{2}|2\rangle - \frac{i}{2}|3\rangle$.

First,

$$\langle\psi|\psi\rangle = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

Now we need to check that the same is true for the QFT, $|\hat{\psi}\rangle$.

$$
\begin{aligned}
|\hat{\psi}\rangle &= \tfrac{1}{4}\textstyle\sum_{k=0}^{3}\left(\sum_{j=0}^{3}\alpha_j e^{\frac{\pi i}{2}jk}\right)|k\rangle \\
&= \begin{cases}
\quad \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{i}{2}+\tfrac{1}{2}-\tfrac{i}{2}\right)|0\rangle & (k=0)\\
+\quad \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{i}{2}e^{\frac{\pi i}{2}}+\tfrac{1}{2}e^{\frac{\pi i}{2}2}-\tfrac{i}{2}e^{\frac{\pi i}{2}3}\right)|1\rangle & (k=1)\\
+\quad \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{i}{2}e^{\frac{\pi i}{2}2}+\tfrac{1}{2}e^{\frac{\pi i}{2}4}-\tfrac{i}{2}e^{\frac{\pi i}{2}6}\right)|2\rangle & (k=2)\\
+\quad \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{i}{2}e^{\frac{\pi i}{2}3}+\tfrac{1}{2}e^{\frac{\pi i}{2}6}-\tfrac{i}{2}e^{\frac{\pi i}{2}9}\right)|3\rangle & (k=3)
\end{cases}\\
&= \begin{cases}
\quad \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{i}{2}+\tfrac{1}{2}-\tfrac{i}{2}\right)|0\rangle\\
+\quad \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{i}{2}i+\tfrac{1}{2}(-1)-\tfrac{i}{2}(-i)\right)|1\rangle\\
+\quad \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{i}{2}(-1)+\tfrac{1}{2}-\tfrac{i}{2}(-1)\right)|2\rangle\\
+\quad \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{i}{2}(-i)+\tfrac{1}{2}(-1)-\tfrac{i}{2}i\right)|3\rangle
\end{cases}\\
&= \begin{cases}
\quad \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{i}{2}+\tfrac{1}{2}-\tfrac{i}{2}\right)|0\rangle\\
+\quad \tfrac{1}{2}\left(\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}\right)|1\rangle\\
+\quad \tfrac{1}{2}\left(\tfrac{1}{2}+\tfrac{i}{2}+\tfrac{1}{2}+\tfrac{i}{2}\right)|2\rangle\\
+\quad \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{1}{2}\right)|3\rangle
\end{cases}\\
&= \tfrac{1-i}{2}|0\rangle+\tfrac{1+i}{2}|2\rangle
\end{aligned}
$$

We find that $|\hat{\psi}\rangle$ is also normal:

$$\langle\hat{\psi}|\hat{\psi}\rangle = \left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right)+\left(\frac{1-i}{2}\right)\left(\frac{1+i}{2}\right) = \frac{2}{4}+\frac{2}{4} = 1$$

■

What is the QFT?

The QFT maps superpositions of N states, $\{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}$, to N states.

A good way to see how the QFT affects states is to write the coefficients as functions of their indices like this:

$$|g\rangle = \sum_{j=0}^{N-1} g_j|j\rangle \equiv \sum_{j=0}^{N-1} g(j)|j\rangle$$

and write its QFT as

$$|G\rangle \equiv \mathcal{F}|g\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\left(\sum_{j=0}^{N-1} g(j)e^{\frac{2\pi i}{N}jk}\right)|k\rangle = \sum_{k=0}^{N-1} G(k)|k\rangle$$

Remember that the QFT maps N states, $\{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}$, to N states. The $e^{\frac{2\pi i}{N}jk}$, "kernel" implies that we should define the j in $g(j)$ as a $mod\,N$ number, $j \in \mathbb{Z}_N$. Modular arithmetic treats the numbers from 0 to $N-1$ like a ring. For example, $2-5 = N-3$ and $(N-4)+7 = 3$.

$$\begin{array}{ccccccccc} |-2\rangle & |-1\rangle & |0\rangle & |1\rangle & \ldots & |N-2\rangle & |N-1\rangle & |N\rangle & |N+1\rangle & |N+2\rangle \\ & & & & \downarrow mod\,N & & & & & \\ |N-2\rangle & |N-1\rangle & |0\rangle & |1\rangle & \ldots & |N-2\rangle & |N-1\rangle & |0\rangle & |1\rangle & |2\rangle \end{array}$$

Mod N dovetails well with the QFT and helps things like this to make sense.

$$\begin{aligned}
\mathcal{F}|g(j+\ell N)\rangle &= \tfrac{1}{\sqrt{N}}\textstyle\sum_{k=0}^{N-1}\left(\sum_{j=0}^{N-1} g(j+\ell N)e^{\frac{2\pi i}{N}jk}\right)|k\rangle \\
&= \tfrac{1}{\sqrt{N}}\textstyle\sum_{k=0}^{N-1}\left(\sum_{j=0}^{N-1} g(j)e^{\frac{2\pi i}{N}(j-\ell N)k}\right)|k\rangle \\
&= \tfrac{1}{\sqrt{N}}\textstyle\sum_{k=0}^{N-1}\left(\sum_{j=0}^{N-1} g(j)e^{\frac{2\pi i}{N}jk}e^{-\frac{2\pi i}{N}\ell N k}\right)|k\rangle \\
&= \tfrac{1}{\sqrt{N}}\textstyle\sum_{k=0}^{N-1}\left(\sum_{j=0}^{N-1} g(j)e^{\frac{2\pi i}{N}jk}\left(e^{-2\pi i}\right)^{\ell k}\right)|k\rangle \\
&= \tfrac{1}{\sqrt{N}}\textstyle\sum_{k=0}^{N-1}\left(\sum_{j=0}^{N-1} g(j)e^{\frac{2\pi i}{N}jk}\right)|k\rangle \\
&= \mathcal{F}|g(j)\rangle
\end{aligned}$$

In other words, $\mathcal{F}|g(j+\ell N)\rangle = \mathcal{F}|g(j)\rangle$ because $g(j+\ell N) = g(j)$ for all j, ℓ.

In what follows, it will be useful to know the following useful identity.

Theorem. $$\boxed{\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}jk} = N\delta_{0,j} = \begin{cases} N & , j = 0 \\ 0 & , j \neq 0 \end{cases}}$$

This is easy to show. For $j \neq 0$,

$$\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}jk} = \sum_{k=0}^{N-1}\left(e^{\frac{2\pi i}{N}j}\right)^k = \frac{e^{\frac{2\pi i}{N}jN}-1}{e^{\frac{2\pi i}{N}j}-1} = \frac{e^{2\pi ij}-1}{e^{\frac{2\pi i}{N}j}-1} = \frac{1-1}{e^{\frac{2\pi i}{N}j}-1} = 0$$

and for $j = 0$,

$$\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}0k} = \sum_{k=0}^{N-1} 1 = N$$

We can actually generalize this a little.

$$\boxed{\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}(tj)k} = N\delta_{0,tj} = \begin{cases} N & ,tj \, mod \, N = 0 \\ 0 & ,tj \, mod \, N \neq 0 \end{cases}}$$

QED

Example

$$|g\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-\frac{2\pi i}{N}tj}|j\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \left[\cos\left(-\frac{2\pi}{N}tj\right) + i\sin\left(-\frac{2\pi}{N}tj\right)\right]|j\rangle$$

Find $|G\rangle = \mathcal{F}|g\rangle$.

$g(j)$ is a simple wave, with angular frequency $2\pi\frac{t}{N}$. The whole point of the QFT is the detection and isolation of states like this.

Applying the definition of the QFT

$$\mathcal{F}|j\rangle \equiv \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}jk}|k\rangle$$

we see that

$$\mathcal{F}|g\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-\frac{2\pi i}{N}tj} \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}jk}|k\rangle\right) = \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{j=0}^{N-1} e^{\frac{2\pi i}{N}j(k-t)}\right)|k\rangle = \frac{1}{N} \sum_{k=0}^{N-1} N\delta_{k,t}|k\rangle = |t\rangle$$

This is what the QFT does; it translates waves, $|g\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-\frac{2\pi i}{N}tj}|j\rangle$, into points, $|G\rangle = |t\rangle$.

■

Example

$$|g\rangle = |t\rangle = \sum_{j=0}^{N-1} \delta_{j,t}|j\rangle$$

Find $|G\rangle = \mathcal{F}|g\rangle$.

This time, $|g\rangle$ is a point. Applying the definition of the QFT

$$\mathcal{F}|j\rangle \equiv \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} jk} |k\rangle$$

we see that

$$\mathcal{F}|t\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} tk} |k\rangle$$

The QFT maps points to waves! $|t\rangle$ gets mapped to a wave with angular frequency $2\pi \frac{t}{N}$.

■

Given the last two examples ($\mathcal{F}$ maps waves to points and points to waves), what's the difference between $\mathcal{F}$ and $\mathcal{F}^{-1} = \mathcal{F}^{\dagger}$? Is $\mathcal{F}$ its own inverse?

$$\begin{aligned}
\mathcal{F}|g\rangle &= \mathcal{F}\left[\textstyle\sum_{j=0}^{N-1} g(j)|j\rangle\right] \\
&= \textstyle\sum_{j=0}^{N-1} g(j)\mathcal{F}|j\rangle \\
&= \textstyle\sum_{j=0}^{N-1} g(j) \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} jk}|k\rangle\right) \\
&= \textstyle\sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} g(j) e^{\frac{2\pi i}{N} jk}\right] |k\rangle
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}^2|g\rangle &= \textstyle\sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} g(j) e^{\frac{2\pi i}{N} jk}\right] \mathcal{F}|k\rangle \\
&= \textstyle\sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} g(j) e^{\frac{2\pi i}{N} jk}\right] \left(\frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} e^{\frac{2\pi i}{N} k\ell}|\ell\rangle\right) \\
&= \textstyle\frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \sum_{\ell=0}^{N-1} g(j) e^{\frac{2\pi i}{N} jk} e^{\frac{2\pi i}{N} k\ell} |\ell\rangle \\
&= \textstyle\frac{1}{N} \sum_{\ell=0}^{N-1} \left[\sum_{j=0}^{N-1} g(j) \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}(j+\ell)k}\right] |\ell\rangle \\
&= \textstyle\frac{1}{N} \sum_{\ell=0}^{N-1} \left[\sum_{j=0}^{N-1} g(j) N \delta_{j,-\ell}\right] |\ell\rangle \\
&= \textstyle\sum_{j=0}^{N-1} g(j) |-j\rangle \\
&= \textstyle\sum_{j=0}^{N-1} g(-j) |j\rangle \\
&= \textstyle\sum_{j=0}^{N-1} g(N-j) |j\rangle
\end{aligned}$$

Evidently,

$$\boxed{\mathcal{F}^2|g(j)\rangle = |g(N-j)\rangle}$$

whereas

$$\mathcal{F}^{\dagger}\mathcal{F}|g(j)\rangle = |g(j)\rangle$$

So the QFT (like any Fourier transform) is very similar to its own inverse; the original $g(j)$ is recovered, just in reverse order, $g(N-j)$.

Example

$$|g(j+h)\rangle = |t\rangle = \sum_{j=0}^{N-1} g(j+h)|j\rangle$$

Find the QFT.

Remember that the coefficients in the QFT are $G(k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} g(j) e^{\frac{2\pi i}{N} jk}$.

$$\begin{aligned}
\mathcal{F}|g(j+h)\rangle &= \textstyle\sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} g(j+h) e^{\frac{2\pi i}{N} jk} \right] |k\rangle \\
&= \textstyle\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} g(j) e^{\frac{2\pi i}{N}(j-h)k} |k\rangle \\
&= \textstyle\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{-\frac{2\pi i}{N} hk} \sum_{j=0}^{N-1} g(j) e^{\frac{2\pi i}{N} jk} |k\rangle \\
&= \textstyle\sum_{k=0}^{N-1} e^{-\frac{2\pi i}{N} hk} G(k) |k\rangle
\end{aligned}$$

Since $\mathcal{F}|t\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} tk} |k\rangle$, we can write this as the product of two QFTs

$$\boxed{\mathcal{F}|g(j+h)\rangle = [\mathcal{F}|-h\rangle]\,[\mathcal{F}|g(j)\rangle]}$$

In other words, shifting $g(j)$ multiplies the QFT with a wave.

■

Example Find the QFT of

$$|\psi\rangle = \sqrt{\frac{r}{N}} \sum_{t=0}^{\frac{N}{r}-1} |\ell + tr\rangle$$

where $\frac{N}{r}$ is an integer.

This is a string of states, spaced r apart and shifted by t. The even spacing implies a simple frequency spectrum and simple QFT.

$$|\hat{\psi}\rangle = QFT|\psi\rangle = \left[\frac{1}{\sqrt{N}} \sum_{j,k=0}^{N-1} e^{\frac{2\pi i}{N} jk} |k\rangle\langle j| \right] \sqrt{\frac{r}{N}} \sum_{t=0}^{\frac{N}{r}-1} |\ell + tr\rangle$$

$$= \frac{\sqrt{r}}{N} \sum_{j,k=0}^{N-1} \sum_{t=0}^{\frac{N}{r}-1} e^{\frac{2\pi i}{N} jk} |k\rangle\langle j|\ell + tr\rangle = \frac{\sqrt{r}}{N} \sum_{k=0}^{N-1} \sum_{t=0}^{\frac{N}{r}-1} e^{\frac{2\pi i}{N}(\ell + tr)k} |k\rangle$$

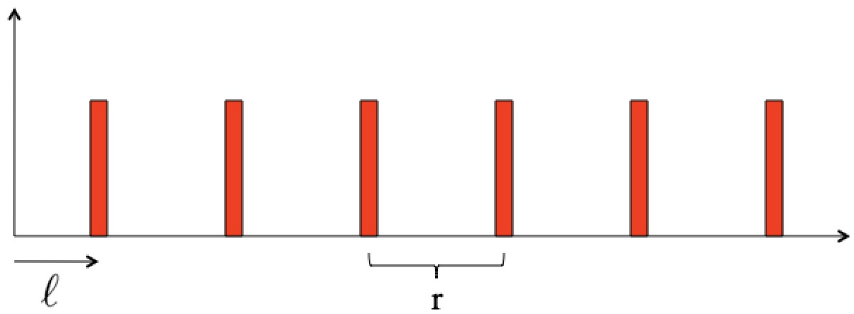

Figure 2: $|\psi\rangle$ is the sum of states spaced r apart and shifted by ℓ.

$$= \frac{\sqrt{r}}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}\ell k}\left(\sum_{t=0}^{\frac{N}{r}-1} e^{\frac{2\pi i r}{N}tk}\right)|k\rangle = \frac{\sqrt{r}}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}\ell k}\left(\frac{N}{r}\delta_{0,k\,mod\,\frac{N}{r}}\right)|k\rangle$$

$k\,mod\,\frac{N}{r} = 0$ means that $k = 0, \frac{N}{r}, 2\frac{N}{r}, \ldots, (r-1)\frac{N}{r}$.

$$= \frac{1}{\sqrt{r}}\sum_{s=0}^{r-1} e^{\frac{2\pi i}{N}\ell s\frac{N}{r}}|s\frac{N}{r}\rangle = \frac{1}{\sqrt{r}}\sum_{s=0}^{r-1} e^{\frac{2\pi i}{r}\ell s}|s\frac{N}{r}\rangle$$

This means that the QFT of a function that's spaced r apart is another function spaced $\frac{N}{r}$ apart.

■

The QFT Circuit

We can write $|k\rangle = |k_1\rangle|k_2\rangle\ldots|k_n\rangle$ where $k = k_1 2^{n-1} + k_2 2^{n-2} + \ldots + k_{n-1}2 + k_n$ and this allows us to get a sense of how to implement a QFT as a circuit by revealing what's actually happening to each individual qubit.

$$
\begin{aligned}
&\mathcal{F}|j\rangle \\
= \quad & \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{\frac{2\pi i}{2^n} jk} |k\rangle \\
= \quad & \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \dots \sum_{k_n=0}^{1} e^{\frac{2\pi i}{2^n} j\left(k_1 2^{n-1} + k_2 2^{n-2} + \ldots + k_{n-1} 2 + k_n\right)} |k_1\rangle |k_2\rangle \dots |k_n\rangle \\
= \quad & \frac{1}{2^{n/2}} \left(\sum_{k_1=0}^{1} e^{\frac{2\pi i}{2^n} jk_1 2^{n-1}} |k_1\rangle\right) \left(\sum_{k_2=0}^{1} e^{\frac{2\pi i}{2^n} jk_2 2^{n-2}} |k_2\rangle\right) \dots \left(\sum_{k_{n-1}=0}^{1} e^{\frac{2\pi i}{2^n} jk_{n-1} 2} |k_{n-1}\rangle\right) \left(\sum_{k_n=0}^{1} e^{\frac{2\pi i}{2^n} jk_n} |k_n\rangle\right) \\
= \quad & \frac{1}{2^{n/2}} \left(\sum_{k_1=0}^{1} e^{2\pi i jk_1 2^{-1}} |k_1\rangle\right) \left(\sum_{k_2=0}^{1} e^{2\pi i jk_2 2^{-2}} |k_2\rangle\right) \dots \left(\sum_{k_{n-1}=0}^{1} e^{2\pi i jk_{n-1} 2^{-n+1}} |k_{n-1}\rangle\right) \left(\sum_{k_n=0}^{1} e^{2\pi i jk_n 2^{-n}} |k_n\rangle\right) \\
= \quad & \frac{1}{2^{n/2}} \prod_{\ell=1}^{n} \left(\sum_{k_\ell=0}^{1} e^{2\pi i jk_\ell 2^{-\ell}} |k_\ell\rangle\right) \\
= \quad & \frac{1}{2^{n/2}} \prod_{\ell=1}^{n} \left(|0\rangle_\ell + e^{2\pi i j 2^{-\ell}} |1\rangle_\ell\right)
\end{aligned}
$$

Something very clever happens in that exponential when we look at the binary expansion $j = j_1 2^{n-1} + j_2 2^{n-2} + \ldots j_{n-1} 2 + j_n$; the integer part[3] of $j 2^{-\ell}$ disappears, since it produces multiples of $e^{2\pi i}$. So as ℓ gets larger, more digits in j are pulled below the decimal point[4] and are retained in the exponential.

$$
\begin{aligned}
= \quad & \frac{1}{2^{n/2}} \left(|0\rangle_1 + e^{2\pi i j 2^{-1}} |1\rangle_1\right) \left(|0\rangle_2 + e^{2\pi i j 2^{-2}} |1\rangle_2\right) \left(|0\rangle_3 + e^{2\pi i j 2^{-3}} |1\rangle_3\right) \dots \\
= \quad & \frac{1}{2^{n/2}} \left(|0\rangle_1 + e^{2\pi i \left(j_n 2^{-1}\right)} |1\rangle_1\right) \left(|0\rangle_2 + e^{2\pi i \left(j_{n-1} 2^{-1} + j_n 2^{-2}\right)} |1\rangle_2\right) \left(|0\rangle_3 + e^{2\pi i \left(j_{n-2} 2^{-1} + j_{n-1} 2^{-2} + j_n 2^{-3}\right)} |1\rangle_3\right) \dots
\end{aligned}
$$

j_1 only affects k_n, j_2 only affects k_{n-1} and k_n, and so on until j_n which makes an appearance in every qubit in $|k\rangle$. So, we change j_1 into k_n using a series of controlled phase gates (see fig. 3)

$$
R_m \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^m}} \end{bmatrix}
$$

then repeat the process for $j_2 \longrightarrow k_{n-1}$ and so on.

H plays the roll of a "self controlled R_1 gate", since for $t = 0, 1$, $H|t\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle + (-1)^t |1\rangle\right] = \frac{1}{\sqrt{2}} \left[|0\rangle + e^{\frac{2\pi i}{2} t} |1\rangle\right]$.

We can see again that the QFT is unitary, every single one of its constituent gates is unitary.

Phase Estimation

Suppose that there's a unitary operator U with eigenstate $|u\rangle$ and eigenvalue

[3] The largest integer less than or equal to $j 2^{-\ell}$.

[4] Technically "binary point", since this is in base two. "Radix point" if you don't want to specify the base.

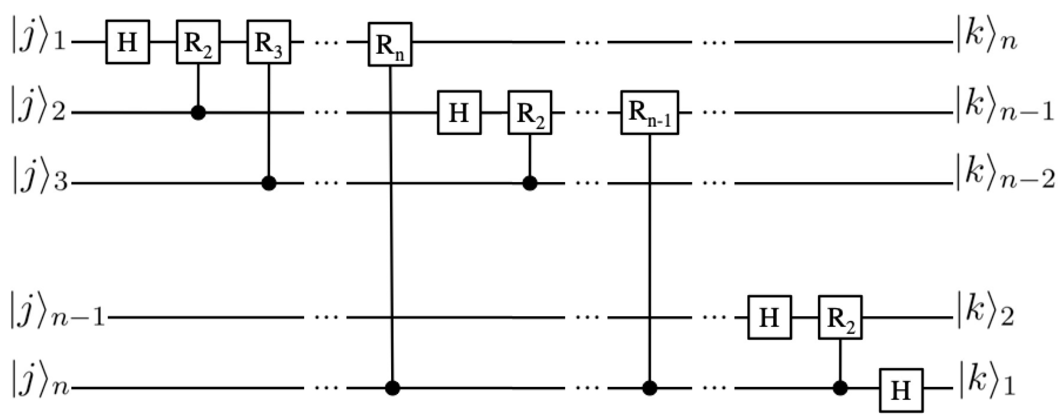

Figure 3: The QFT circuit that maps $|j\rangle \longrightarrow 2^{-\frac{n}{2}} \sum_{k=0}^{2^n-1} e^{\frac{2\pi i}{2^n} jk} |k\rangle$.

$$\boxed{\lambda = e^{2\pi i \varphi}}$$

where $\varphi \in [0,1)$. We know that we can write λ in this form because for unitary operators $|\lambda| = 1$.

The idea of the phase estimation algorithm is that U and whatever process prepares $|u\rangle$ are "oracles"; black box functions that we don't have access to. We have two registers (sets of qubits). The first register begins with an t qubits initialized to $|0\rangle^{\otimes t}$, where t is how many binary digits of φ we want to know. That is, we hope to learn φ to within $\frac{1}{2^t}$. The second register is initialized to $|u\rangle$, which involves some unknown (and unimportant) number of qubits.

The initial state of the system is

$$|0\rangle^{\otimes t}|u\rangle = |0\rangle_1|0\rangle_2 \dots |0\rangle_t|u\rangle$$

and after the bank of Hadamard gates the system is

$$\frac{1}{2^{t/2}} \left[|0\rangle_1 + |1\rangle_1\right] \left[|0\rangle_2 + |1\rangle_2\right] \dots \left[|0\rangle_t + |1\rangle_t\right] |u\rangle$$

or more succinctly, by using the binary expansion notation $|x\rangle = |x_1\rangle|x_2\rangle \dots |x_t\rangle$,

$$\frac{1}{2^{t/2}} \sum_{x=0}^{2^t-1} |x\rangle|u\rangle$$

These controlled U gates do the following:

$$CU^k|0\rangle|u\rangle = |0\rangle|u\rangle \qquad\qquad CU^k|1\rangle|u\rangle = \left(e^{2\pi i\varphi}\right)^k |1\rangle|u\rangle = e^{2\pi i k\varphi}|1\rangle|u\rangle$$

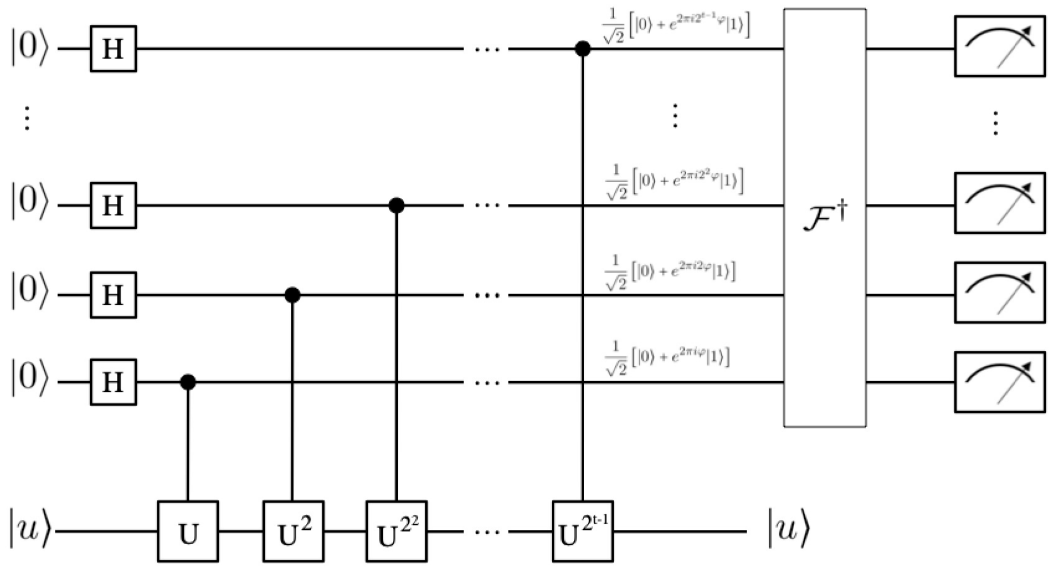

Figure 4: The phase-estimation circuit.

Even with all the other qubits floating around, that coefficient stays stuck to the appropriate state of the appropriate qubit. For example, look at the first CU gate, which is controlled by the t qubit:

$$\begin{aligned}
& CU\left[\tfrac{1}{2^{t/2}}\left[|0\rangle_1+|1\rangle_1\right]\ldots\left[|0\rangle_{t-1}+|1\rangle_{t-1}\right]\left[|0\rangle_t+|1\rangle_t\right]|u\rangle\right] \\
= & \tfrac{1}{2^{t/2}}\left[|0\rangle_1+|1\rangle_1\right]\ldots\left[|0\rangle_{t-1}+|1\rangle_{t-1}\right]CU\left[|0\rangle_t+|1\rangle_t\right]|u\rangle \\
= & \begin{cases} & \tfrac{1}{2^{t/2}}\left[|0\rangle_1+|1\rangle_1\right]\ldots\left[|0\rangle_{t-1}+|1\rangle_{t-1}\right]CU|0\rangle_t|u\rangle \\ + & \tfrac{1}{2^{t/2}}\left[|0\rangle_1+|1\rangle_1\right]\ldots\left[|0\rangle_{t-1}+|1\rangle_{t-1}\right]CU|1\rangle_t|u\rangle \end{cases} \\
= & \begin{cases} & \tfrac{1}{2^{t/2}}\left[|0\rangle_1+|1\rangle_1\right]\ldots\left[|0\rangle_{t-1}+|1\rangle_{t-1}\right]|0\rangle_t|u\rangle \\ + & e^{2\pi i\varphi}\tfrac{1}{2^{t/2}}\left[|0\rangle_1+|1\rangle_1\right]\ldots\left[|0\rangle_{t-1}+|1\rangle_{t-1}\right]|1\rangle_t|u\rangle \end{cases} \\
= & \tfrac{1}{2^{t/2}}\left[|0\rangle_1+|1\rangle_1\right]\ldots\left[|0\rangle_{t-1}+|1\rangle_{t-1}\right]\left[|0\rangle_t+e^{2\pi i\varphi}|1\rangle_t\right]|u\rangle
\end{aligned}$$

After all the controlled U^{2^k} gates the state of the system is

$$\frac{1}{2^{t/2}}\left[|0\rangle_1+e^{2\pi i\left(2^{t-1}\varphi\right)}|1\rangle_1\right]\left[|0\rangle_2+e^{2\pi i\left(2^{t-2}\varphi\right)}|1\rangle_2\right]\ldots\left[|0\rangle_t+e^{2\pi i\varphi}|1\rangle_t\right]|u\rangle$$

or using the binary expansion notation and noting that $x = x_1 2^{t-1} + x_2 2^{t-2} + \ldots + x^{t-1}2 + x_t$,

$$\frac{1}{2^{t/2}}\sum_{x=0}^{2^t-1} e^{2\pi i x\varphi}|x\rangle|u\rangle$$

We can ignore the second register, since we always know exactly what it is. Notice that the coefficients of the first register follow a very simple pattern and are described by a single frequency. Writing the coefficients using Euler's identity

$$e^{2\pi i x\varphi} = \cos(2\pi\varphi x) + i\sin(2\pi\varphi x)$$

we see that φ is the frequency ($2\pi\varphi$ is the angular frequency). Because the frequency spectrum is so simple, the QFT is an obvious tool to use. While $\frac{1}{2^{t/2}}\sum_{x=0}^{2^t-1} e^{2\pi i x\varphi}|x\rangle$ is evenly spread out over all the $|x\rangle$ states, we can expect the QFT to be highly concentrated.

If we write the first register as

$$\frac{1}{2^{t/2}}\sum_{x=0}^{2^t-1} e^{\frac{2\pi i}{2^t}x\left(2^t\varphi\right)}|x\rangle$$

then it looks suspiciously like the QFT of $|2^t\varphi\rangle$! In fact, if φ has a binary expansion no longer than t digits, then that's exactly what this is. We'll ignore the second register, $|u\rangle$, since we know exactly what it is and there's nothing more to gain from it.

The inverse quantum Fourier transform of the first register yields

$$\mathcal{F}^\dagger\left[\frac{1}{2^{t/2}}\sum_{x=0}^{2^t-1} e^{\frac{2\pi i}{2^t}x\left(2^t\varphi\right)}|x\rangle\right] = \frac{1}{2^t}\sum_{x,y=0}^{2^t-1} e^{-\frac{2\pi i}{2^t}xy}e^{\frac{2\pi i}{2^t}x\left(2^t\varphi\right)}|y\rangle = \sum_{y=0}^{2^t-1}\left[\frac{1}{2^t}\sum_{x=0}^{2^t-1} e^{\frac{2\pi i}{2^t}x\left(2^t\varphi-y\right)}\right]|y\rangle$$

As messy as it is, the coefficient in front of every $|y\rangle$ is a geometric series. The probability of

$$p(|y\rangle) = \left|\frac{1}{2^t}\sum_{x=0}^{2^t-1} e^{\frac{2\pi i}{2^t}x\left(2^t\varphi-y\right)}\right|^2 = \frac{1}{4^t}\left|\frac{1-e^{2\pi i\left(2^t\varphi-y\right)}}{1-e^{\frac{2\pi i}{2^t}\left(2^t\varphi-y\right)}}\right|^2$$

This is as simple as this is going to get, but we can bound the numerator from below and denominator from above to find a lower bound for $p(|k\rangle)$. Assume that

$$\left|2^t\varphi - y\right| \leq \frac{1}{2}$$

meaning that $\frac{y}{2^t}$ is the closest t-digit binary approximation to φ; if you round off $2^t\varphi$, you get y.

Note that $\left|1-e^{i\theta}\right| = 2\left|\sin\left(\frac{\theta}{2}\right)\right| = 2\sin\left(\frac{|\theta|}{2}\right)$ and that for $x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ we have $\frac{2}{\pi}|x| \leq \sin|x| \leq |x|$. So when $\theta \in [-\pi,\pi]$

$$\frac{2}{\pi}|\theta| \leq \left|1-e^{i\theta}\right| \leq |\theta|$$

Therefore, since $2\pi\left(2^t\varphi - y\right) \leq 2\pi\left(\frac{1}{2}\right) = \pi$,

$$p(|y\rangle) = \frac{1}{4^t}\left|\frac{1-e^{2\pi i\left(2^t\varphi-y\right)}}{1-e^{\frac{2\pi i}{2^t}\left(2^t\varphi-y\right)}}\right|^2 \geq \frac{4}{\pi^2 4^t}\left|\frac{2\pi\left(2^t\varphi-y\right)}{1-e^{\frac{2\pi i}{2^t}\left(2^t\varphi-y\right)}}\right|^2 \geq \frac{4}{\pi^2 4^t}\left|\frac{2\pi\left(2^t\varphi-y\right)}{\frac{2\pi}{2^t}\left(2^t\varphi-y\right)}\right|^2 = \frac{4}{\pi^2}$$

$$\boxed{p(best\ approximation) \geq \frac{4}{\pi^2} \approx 0.405}$$

This means that the phase estimation circuit yields the best t-digit binary approximation for φ at least 40% of the time. So we can't expect this to work perfectly the first time, but if you repeat the procedure several times, the result you get most (and at least 40% of the total after many runs) is the best approximation of $2^t\varphi$, and therefore $e^{2\pi i\left(\frac{y}{2^t}\right)}$ is the best approximation of $\lambda = e^{2\pi i\varphi}$.

Exercises

#1) Patterns.

Find the QFT of each given state in the space with the basis $\{|0\rangle, |1\rangle, \ldots, |7\rangle\}$.

Hint: It's will be quicker to do these calculations if you use $\omega = e^{\frac{\pi}{4}i}$*, instead of writing out the exponential every time, and if you keep in mind that sums of numbers equally-spaced around the unit circle are zero (e.g.,* $\omega^0 + \omega^2 + \omega^4 + \omega^6 = 1 + i + (-1) + (-i) = 0$*).*

a)

$$|\xi\rangle = \frac{1}{\sqrt{8}}|0\rangle + \frac{1}{\sqrt{8}}|1\rangle + \frac{1}{\sqrt{8}}|2\rangle + \frac{1}{\sqrt{8}}|3\rangle + \frac{1}{\sqrt{8}}|4\rangle + \frac{1}{\sqrt{8}}|5\rangle + \frac{1}{\sqrt{8}}|6\rangle + \frac{1}{\sqrt{8}}|7\rangle$$

b)

$$|\eta\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|4\rangle + \frac{1}{2}|6\rangle$$

c)

$$|\mu\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|4\rangle$$

c)

$$|\gamma\rangle = |0\rangle$$

e) What patterns do you notice?

#2) Unitarity.

Show that the QFT is unitary by direct calculation.

#3) Estimaser.

Assume you've run the phase estimation algorithm several times and gotten the results:

$$|0\rangle|1\rangle|1\rangle|0\rangle|1\rangle|1\rangle|0\rangle|0\rangle$$
$$|0\rangle|1\rangle|1\rangle|0\rangle|1\rangle|1\rangle|0\rangle|1\rangle$$
$$|0\rangle|1\rangle|1\rangle|0\rangle|1\rangle|0\rangle|0\rangle|1\rangle$$
$$|1\rangle|1\rangle|1\rangle|0\rangle|1\rangle|1\rangle|0\rangle|0\rangle$$
$$|1\rangle|1\rangle|0\rangle|0\rangle|1\rangle|1\rangle|0\rangle|0\rangle$$
$$|0\rangle|1\rangle|1\rangle|0\rangle|1\rangle|0\rangle|0\rangle|1\rangle$$
$$|1\rangle|1\rangle|1\rangle|0\rangle|1\rangle|0\rangle|0\rangle|0\rangle$$

What is your best guess for the eigenvalue, λ?

#4) N to Beam Up.

Teleportation doesn't just work for qubits, it works for "qudits" (quantum systems of d dimensions) as well. Suppose we want to teleport an N-dimensional quantum state, $|\psi\rangle_a = \sum_{k=0}^{N-1} \alpha_k |k\rangle_a$, from space A to C. Using entangled states of the form

$$|\chi_{s,t}\rangle_{bc} \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{\frac{2\pi i}{N} sj} |j\rangle_b |j \oplus t\rangle_c$$

These will take the place of the Bell states for our more general teleportation procedure. Here $j \oplus t$ is addition mod N. Notice that $e^{\frac{2\pi i}{N}(a \oplus b)} = e^{\frac{2\pi i}{N}(a+b)}$ since multiples of N can be ignored in this exponent: $e^{\frac{2\pi i}{N}(a\oplus b)} = e^{\frac{2\pi i}{N}(a+b+\ell N)} = e^{\frac{2\pi i}{N}(a+b)} e^{\ell 2\pi i}$.

(a) Show that

$$\langle \chi_{s,t} | \chi_{s',t'} \rangle = \delta_{s,s'} \delta_{t,t'}$$

b) We start with the state

$$|\psi\rangle_a |\chi_{s,t}\rangle_{bc}$$

Find the possible results from a measurement of $A \otimes B$ in the $\{|\chi_{s,t}\rangle\}$ basis. In other words, calculate:

$$\langle \chi_{x,y}|_{ab} \left(|\psi\rangle_a |\chi_{s,t}\rangle_{bc} \right)$$

c) Using the result of the measurement in part b, find the unitary transformation that will reconstruct the state $|\psi\rangle$ on C. This unitary operation will depend on the indices of both the initial entangled state, $|\chi_{s,t}\rangle_{bc}$, and the result of the measurement, $|\chi_{x,y}\rangle_{ab}$.

(Hint: First ask yourself "How do I fix the ket?" then ask yourself "How do I fix the phase?")