QI Lecture 16

Quantum Measurements

Projective Measurements

The measurements we've considered in the class so far have been "projective measurements" meaning that the measurement operators are projections.[1] We'll first review what we know by summing up the properties of these measurements with a few postulates. We define an "observable" as

$$M = \sum_m m P_m$$

and will assume we're measuring a state described either[2] as $|\psi\rangle$ or ρ using this observable.

[1] It's a good name.

[2] We can use either kets or density matrices for pure states, but for mixed states we must use density matrices.

1) The measurement operators are a set, $\{P_m\}$, that span the entire space.

$$\sum_m P_m P_m^\dagger = \sum_m P_m = I$$

This is just another way of saying that no matter what state you measure, you'll always get a result.

2) The measurement operations, P_m, for projective measurements are projections

$$P_j P_k = \delta_{jk} P_j$$

which also implies that the only possible eigenvalues of these operations are 0 and 1.

3) A measurement yields <u>one</u> eigenvalue, m, corresponding to one projection operator, P_m.

4) The probability of result m is

$$p(m) = \left\| P_m|\psi\rangle \right\|^2 = \langle\psi|P_m^\dagger P_m|\psi\rangle = \langle\psi|P_m|\psi\rangle \qquad p(m) = Tr\left[P_m \rho P_m^\dagger\right] = Tr\left[P_m \rho\right]$$

5) Following a measurement with result m the state of the system is

$$|\psi_m\rangle = \frac{P_m|\psi\rangle}{\sqrt{p(m)}} = \frac{P_m|\psi\rangle}{\sqrt{\langle\psi|P_m^\dagger P_m|\psi\rangle}} \qquad \rho_m = \frac{P_m \rho P_m^\dagger}{p(m)} = \frac{P_m \rho P_m^\dagger}{Tr\left[P_m \rho P_m^\dagger\right]}$$

This is a clean "cookie-cutter" removal of every part of the original state consistent with the measurement, re-normalized.

6) An unreported measurement produces a mixed state

$$\rho' = \sum_m p(m)|\psi_m\rangle\langle\psi_m| = \sum_m P_m|\psi\rangle\langle\psi|P_m^\dagger \qquad \rho' = \sum_m p(m)\rho_m = \sum_m P_m \rho_m P_m^\dagger$$

because we're left with an ensemble of states where $p(|\psi_m\rangle) = p(m)$. One effect of this is to destroy any coherence that exists between mutually exclusive measurements. Although we may not be aware of what measurement occurred, we know (from postulate #3) that there was only one and therefore the state cannot be in a superposition that spans multiple results.

All of the daggers above are completely pointless, since $P_m^\dagger = P_m$, and many of the projections are redundant, since $P_m^2 = P_m$. We're writing it this way so that the generalized measurements below will seem more familiar and reasonable.

Positive Operator Valued Measures (POVMs)

There's nothing about quantum measurements that requires us to use projection operators to describe them.We'll drop postulate #2 and generalize to a new set of operators

$$P_m \longrightarrow A_m \qquad\qquad P_m^\dagger P_m \longrightarrow A_m^\dagger A_m = \Pi_m$$

$\{A_m\}$ are the "**detection operators**" and $\{\Pi_m\}$ are the "**elements of the POVM**".

For projective operators we can quickly simplify many of the postulates, but with these new operations we no longer have that option. For example,

$$p(m) = Tr\left[P_m \rho P_m^\dagger\right] = Tr\left[P_m^\dagger P_m \rho\right] = Tr\left[P_m P_m \rho\right] = Tr\left[P_m \rho\right]$$

and now

$$p(m) = Tr\left[A_m \rho A_m^\dagger\right] = Tr\left[A_m^\dagger A_m \rho\right] = Tr\left[\Pi_m \rho\right]$$

We find that A_m is important for figuring out what the impact of the measurement is on the initial state and Π_m is important for everything else.

Take a look at postulate #4 above. The positivity of probabilities implies that

$$0 \leq p(m) = \langle\psi|A_m^\dagger A_m|\psi\rangle = \langle\psi|\Pi_m|\psi\rangle$$

for all $|\psi\rangle$, which defines Π_m as a positive operator. Moreover, the normalization of probabilities implies a new postulate #1:

$$1 = \sum_m p(m) = \sum_m \langle\psi|\Pi_m|\psi\rangle = \langle\psi|\left(\sum_m \Pi_m\right)|\psi\rangle \qquad \Rightarrow \qquad \sum_m \Pi_m = I$$

1) The POVM is a set of operators, $\{\Pi_m\}$, that partition the identity operator on the space

$$\sum_m A_m A_m^\dagger = \sum_m \Pi_m = I$$

2) The elements of the POVM, Π_m, are positive operators that can be expressed in terms of detection operators, $\Pi_m = A_m^\dagger A_m$, where the detection operators are not in general hermitian ($A^\dagger \neq A$). Conversely, the detection operators can be expressed as $A_m = U_m\sqrt{\Pi_m}$, where U_m is some unitary operator.

3) A measurement yields one of the alternatives corresponding to an element of the POVM, which we index using m.

4) The probability of result m is

$$p(m) = \langle\psi|A_m^\dagger A_m|\psi\rangle = \langle\psi|\Pi_m|\psi\rangle \qquad p(m) = Tr\left[A_m\rho A_m^\dagger\right] = Tr\left[\Pi_m\rho\right]$$

5) Following a measurement with result m the state of the system is

$$|\psi_m\rangle = \frac{A_m|\psi\rangle}{\sqrt{p(m)}} = \frac{A_m|\psi\rangle}{\sqrt{\langle\psi|\Pi_m|\psi\rangle}} \qquad \rho_m = \frac{A_m\rho A_m^\dagger}{p(m)} = \frac{A_m\rho A_m^\dagger}{Tr\left[A_m\rho A_m^\dagger\right]} = \frac{A_m\rho A_m^\dagger}{Tr\left[\Pi_m\rho\right]}$$

The possible unitary operator in $A_m = U_m\sqrt{\Pi_m}$ allows for a huge amount of flexibility in designing measurements.

6) An unreported measurement produces a mixed state

$$\rho' = \sum_m A_m|\psi\rangle\langle\psi|A_m^\dagger \qquad \rho' = \sum_m p(m)\rho_m = \sum_m A_m\rho_m A_m^\dagger$$

Note that projective measures are a special case of POVMs, since when $A_m = P_m$ all of the postulates for POVMs hold. There are however major differences.

For projective measures there is a limit to the number of possible results. The "smallest" that a projection can be is one-dimensional, $P_k = |k\rangle\langle k|$, and since $P_j P_k = \delta_{jk}P_k$ and $\sum_k P_k = I$,

There are at most as many projection operators in $\{P_m\}$ as there are dimensions in the space.

However, POVMs do not have this restriction.

Neumark's Theorem[3]

Example We've seen before that if spaces A and B share an entangled state, then a (reported) measurement in one space affects the state in the other. This gives us a new way to execute measurements on states $|\psi\rangle$. Suppose that Alice and Bob share the (not maximally) entangled state

$$|\psi\rangle = \frac{|+\rangle_a|0\rangle_b + |1\rangle_a|1\rangle_b}{\sqrt{2}}$$

and that Bob makes a measurement in the $\{|0\rangle_a, |1\rangle_a\}$ basis and announces result. Each with a probability of one half, Alice's qubit after this measurement will be

$$|\psi^{(0)}\rangle_a = |+\rangle_a \qquad\qquad |\psi^{(1)}\rangle_a = |1\rangle_a$$

Notice that this set of results would be impossible for Alice to do with a projective measurement on her own. Projective measurements are always orthogonal, so she could do a measurement in either the $\{|0\rangle_a, |1\rangle_a\}$ or $\{|+\rangle_a, |-\rangle_a\}$, but $\{|+\rangle_a, |1\rangle_a\}$ is not a measurement that can be performed with a projective measurement. This is a job for POVMs or, it seems, entanglement with an ancillary space followed by a measurement in that space.

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More generally, we take an arbitrary state $|\psi\rangle_a \in A$ and attach an ancilla space B with basis $\{|m\rangle_b\}$ (this will be the measurement basis) and an initially separate state $|\phi\rangle_b$, so we begin with the separable state $|\psi\rangle_a|\phi\rangle_b$.

The application of a unitary operator to this composite system, $U_{ab}|\psi\rangle_a|\phi\rangle_b$, entangles it so that the probability of making a projective measurement in B and seeing $|m\rangle_b$ is

$$p(m) = \|(I_a \otimes |m\rangle_b\langle m|_b)\, U_{ab}|\psi\rangle_a|\phi\rangle_b\|^2 = \langle\psi|_a\langle\phi|_b U_{ab}\,(I_a \otimes |m\rangle_b\langle m|_b)\, U_{ab}|\psi\rangle_a|\phi\rangle_b$$

and the resulting state is

$$\frac{|m\rangle_b\langle m|_b U_{ab}|\psi\rangle_a|\phi\rangle_b}{\sqrt{p(m)}}$$

We can write this more compactly by defining the detection operator A_m as

$$\boxed{A_m|\psi\rangle_a \equiv \langle m|_b U_{ab}|\psi\rangle_a|\phi\rangle_b}$$

[3] Also spelled "Naimark", because Soviets don't need Roman letters.

so that

$$p(m) = \langle\psi|_a A_m^\dagger A_m |\psi\rangle_a$$

and the resulting state is

$$\frac{A_m|\psi\rangle_a|m\rangle_b}{\sqrt{p(m)}}$$

or ignoring space B[4]

$$\frac{A_m|\psi\rangle_a}{\sqrt{p(m)}}$$

A_m serves the roll of the measurement operator in space A. While this operator is defined entirely in A, it's dependent on U_{ab}, $|m\rangle_b$, and $|\phi\rangle_b$. Take a moment to consider the notation in the definition of A_m; there's a bra and ket for space B but only a ket for A, which means that this is a quantum state (a ket) in A.

Neumark's theorem says that these two ideas are essentially equivalent. The effect of entangling and then measuring an ancilla space produce detection operators, $\{A_m\}$, and a POVM, $\{\Pi_m = A_m^\dagger A_m\}$, and conversely, a POVM can be performed by entangling and measuring an ancilla space.

First, to show that "entangle $A \otimes B$ and measure B" is a POVM on A.
For <u>any</u> $|\psi\rangle_a$

$$1 = \sum_m p(m) = \sum_m \langle\psi|_a A_m^\dagger A_m |\psi\rangle_a = \langle\psi|_a \left(\sum_m A_m^\dagger A_m\right) |\psi\rangle_a$$

which implies that

$$\sum_m A_m^\dagger A_m = I$$

Moreover, the effect on $|\psi\rangle \in A$ of observing $|m\rangle \in B$ is

$$|\psi\rangle_a \xrightarrow{m} \frac{A_m|\psi\rangle_a}{\sqrt{p(m)}} = \frac{A_m|\psi\rangle_a}{\sqrt{\langle\psi|_a A_m^\dagger A_m |\psi\rangle_a}}$$

Consulting the postulates of POVMs we see that $\left\{A_m^\dagger A_m\right\}$ form a POVM.

[4]This is not a new thing. Since we haven't been keeping track of the state of the observer (so far) we've actually been ignoring the "pointer state", $|m\rangle_b$, for the entire class so far. You can think of space B as you, personally, or whatever apparatus is doing the measurement.

Now to show the converse; that a POVM on A can be expressed through "entangling $A \otimes B$ and measuring B". Specifically, we need to show that given a set of detection operators $\{A_m\}$ such that $I = \sum_m A_m^\dagger A_m$ (a POVM), can we define a space B, a state $|\phi\rangle_b \in B$, and a unitary operator U_{ab} such that $A_m|\psi\rangle_a = \langle m|_b U_{ab}|\psi\rangle_a|\phi\rangle_b$ for all $|\psi\rangle_a \in A$.

It turns out that this is pretty easy because we can absorb all of the arbitrary choices into U_{ab}. Chose an M-dimensional space B with arbitrary basis $\{|m\rangle_b\}$, where M is the desired number of measurement results, and some arbitrary (but fixed) state $|\phi\rangle_b \in B$.

Now define U_{ab} using

$$U_{ab}|\psi\rangle_a|\phi\rangle_b \equiv \sum_m A_m|\psi\rangle_a|m\rangle_b$$

This adheres to the definition for A_m, since

$$\langle m|_b U_{ab}|\psi\rangle_a|\phi\rangle_b = \langle m|_b \left(\sum_k A_k|\psi\rangle_a|k\rangle_b\right) = A_m|\psi\rangle_a$$

We don't yet know that U_{ab} is unitary[5], so to check we need to show that it preserves inner products; the defining characteristic of unitary operations.

$$\begin{aligned}
\langle\eta|_a\langle\phi|_b U_{ab}^\dagger U_{ab}|\psi\rangle_a|\phi\rangle_b &= \left(\textstyle\sum_j \langle\eta|_a A_j^\dagger \otimes \langle j|_b\right)\left(\textstyle\sum_k A_k|\psi\rangle_a \otimes |k\rangle_b\right) \\
&= \textstyle\sum_{jk}\langle\eta|_a A_j^\dagger A_k|\psi\rangle_a\langle j|k\rangle_b \\
&= \textstyle\sum_k \langle\eta|_a A_k^\dagger A_k|\psi\rangle_a \\
&= \langle\eta|_a\left(\textstyle\sum_k A_k^\dagger A_k\right)|\psi\rangle_a \\
&= \langle\eta|\psi\rangle_a
\end{aligned}$$

Since $\langle\eta|\psi\rangle_a = [\langle\eta|_a\langle\phi|_b][|\psi\rangle_a|\phi\rangle_b]$ we have that U_{ab} is unitary on the subspace spanned by $\{|k\rangle_a \otimes |\phi\rangle_b\}_k$. In other words, it's unitary when the state in B is $|\phi\rangle$. We haven't actually bothered to define U_{ab} on the full space $A \otimes B$, so let's do that now. If $|\phi^\perp\rangle_b$ is any state orthogonal to $|\phi\rangle_b$, then

$$U_{ab}|\psi\rangle_a|\phi\rangle_b \equiv \sum_m A_m|\psi\rangle_a|m\rangle_b \qquad\qquad U_{ab}|\psi\rangle_a|\phi^\perp\rangle_b \equiv |\psi\rangle_a|\phi^\perp\rangle_b$$

Defining U_{ab} to be the identity on the space orthogonal to $|\phi\rangle_b$ makes it unitary on $A \otimes B$ in general. Clearly $\langle\eta|_a\langle\phi_1^\perp|_b U_{ab}^\dagger U_{ab}|\psi\rangle_a|\phi_2^\perp\rangle_b = \langle\eta|_a\langle\phi_1^\perp|_b|\psi\rangle_a|\phi_2^\perp\rangle_b$ and because

$$U_{ab}|\psi\rangle_a|\phi^\perp\rangle_b = |\psi\rangle_a|\phi^\perp\rangle_b \quad\Rightarrow\quad U_{ab}^\dagger U_{ab}|\psi\rangle_a|\phi^\perp\rangle_b = U_{ab}^\dagger|\psi\rangle_a|\phi^\perp\rangle_b \quad\Rightarrow\quad |\psi\rangle_a|\phi^\perp\rangle_b = U_{ab}^\dagger|\psi\rangle_a|\phi^\perp\rangle_b$$

we have that

[5] Even though the "U" notation is so suggestive.

$$\langle\eta|_a\langle\phi|_b U_{ab}^\dagger U_{ab}|\psi\rangle_a|\phi^\perp\rangle_b = \langle\eta|_a\langle\phi|_b U_{ab}^\dagger|\psi\rangle_a|\phi^\perp\rangle_b = \langle\eta|_a\langle\phi|_b|\psi\rangle_a|\phi^\perp\rangle_b$$

This proves Neumark's Theorem.

Any POVM on space A, $\{A_m^\dagger A_m\}$, is equivalent to a projection measurement on $A\otimes B$. This can be realized by expressing $A_m|\psi\rangle_a \equiv \langle m|_b U_{ab}|\psi\rangle_a|\phi\rangle_b$ for appropriately chosen U_{ab}, $|\phi\rangle_b$ and $\{|m\rangle_b\}$, and then executing a projective measurement on B in the measurement basis $\{|m\rangle_b\}$.

Unambiguous State Discrimination

In unambiguous state discrimination we'd like to be able to identify states as accurately as possible, without ever mistaking one state for another. Assume that we have an ensemble composed of two non-orthogonal states, $|\psi_1\rangle$ and $|\psi_2\rangle$ such that $\langle\psi_1|\psi_2\rangle \neq 0$, and the density matrix of this ensemble is

$$\rho = r_1|\psi_1\rangle\langle\psi_1| + r_2|\psi_2\rangle\langle\psi_2|$$

This is the situation where Alice is sending Bob the states $|\psi_1\rangle$ and $|\psi_2\rangle$ with probabilities r_1 and r_2 and it's up to Bob to figure out which state was sent without mistaking one for the other.

Unfortunately, it isn't possible to perfectly distinguish non-orthogonal states. That is, there is no POVM $\{\Pi_m\}$ such that

$$\langle\psi_1|\Pi_1|\psi_1\rangle = \langle\psi_2|\Pi_2|\psi_2\rangle = 1 \qquad \langle\psi_1|\Pi_2|\psi_1\rangle = \langle\psi_2|\Pi_1|\psi_2\rangle = 0$$

We can see that this is the case by first noting that

$$\langle\psi_j|\Pi_k|\psi_j\rangle = \langle\psi_j|A_k^\dagger A_k|\psi_j\rangle = \|A_k|\psi_j\rangle\|^2$$

and therefore

$$\langle\psi_j|\Pi_k|\psi_j\rangle = 0 \quad\Rightarrow\quad A_k|\psi_j\rangle = 0 \quad\Rightarrow\quad \Pi_k|\psi_j\rangle = 0 \text{ and } \langle\psi_j|\Pi_k = 0$$

If $\{\Pi_1,\Pi_2\}$ is a POVM that's unambiguous in the sense that $\langle\psi_1|\Pi_2|\psi_1\rangle = \langle\psi_2|\Pi_1|\psi_2\rangle = 0$, then the POVM (as always) partitions the identity and

$$\begin{aligned}
\Pi_1 + \Pi_2 &= I \\
\langle\psi_1|\left[\Pi_1 + \Pi_2\right]|\psi_2\rangle &= \langle\psi_1|I|\psi_2\rangle \\
\langle\psi_1|\Pi_1|\psi_2\rangle + \langle\psi_1|\Pi_2|\psi_2\rangle &= \langle\psi_1|\psi_2\rangle \\
\langle\psi_1|0+0|\psi_2\rangle &= \langle\psi_1|\psi_2\rangle \\
0 &= \langle\psi_1|\psi_2\rangle
\end{aligned}$$

This is a contradiction, since we assume that $|\psi_1\rangle$ and $|\psi_2\rangle$ are non-orthogonal. Working with POVMs means that we can introduce a third measurement operator, Π_0, to the two dimensional space $Span\{|\psi_1\rangle, |\psi_2\rangle\}$. Π_0 is an "error" operator that gives us enough wiggle room to keep the measurement unambiguous, without being perfect.

Define

$$\langle\psi_1|\Pi_1|\psi_1\rangle = p_1 \qquad \langle\psi_2|\Pi_2|\psi_2\rangle = p_2 \qquad \langle\psi_1|\Pi_2|\psi_1\rangle = \langle\psi_2|\Pi_1|\psi_2\rangle = 0$$

$$\langle\psi_1|\Pi_0|\psi_1\rangle = 1 - p_1 = q_1 \qquad \langle\psi_2|\Pi_0|\psi_2\rangle = 1 - p_2 = q_2$$

so that p_j is the probability of successfully detecting $|\psi_j\rangle$ and q_j is the failure probability for $|\psi_j\rangle$. Unambiguity dictates that

$$\begin{aligned} \Pi_1 &= c_1|\psi_2^\perp\rangle\langle\psi_2^\perp| \\ \Pi_2 &= c_2|\psi_1^\perp\rangle\langle\psi_1^\perp| \end{aligned}$$

where $|\psi_j^\perp\rangle \in Span\{|\psi_1\rangle, |\psi_2\rangle\}$ and $\langle\psi_j|\psi_j^\perp\rangle = 0$ uniquely determines $|\psi_j^\perp\rangle$ up to complex phase. We can nail it down up to sign by declaring that $\langle\psi_1|\psi_2^\perp\rangle, \langle\psi_2|\psi_1^\perp\rangle \in \mathbb{R}$.

Plugging this into the definitions we're using for the POVM we find that

$$p_1 = \langle\psi_1|\Pi_1|\psi_1\rangle = c_1\left|\langle\psi_2^\perp|\psi_1\rangle\right|^2 \qquad p_2 = \langle\psi_2|\Pi_2|\psi_2\rangle = c_2\left|\langle\psi_1^\perp|\psi_2\rangle\right|^2$$

and making the declaration that

$$\cos(\theta) \equiv |\langle\psi_1|\psi_2\rangle| \qquad \sin(\theta) \equiv \left|\langle\psi_1^\perp|\psi_2\rangle\right| = \left|\langle\psi_1|\psi_2^\perp\rangle\right|$$

we can write

$$\Pi_1 = \frac{p_1}{\sin^2(\theta)}|\psi_2^\perp\rangle\langle\psi_2^\perp| \qquad \Pi_2 = \frac{p_2}{\sin^2(\theta)}|\psi_1^\perp\rangle\langle\psi_1^\perp|$$

So we have expressions for Π_1 and Π_2 in terms of (the yet to be determined) p_1 and p_2. We already know what Π_0 is, because the POVM partitions the identity:

$$\Pi_0 = I - \Pi_1 - \Pi_2$$

We know that all three elements of the POVM must be positive operators, and that puts restrictions on the possible values of p_j, q_j. In the homework you'll derive that the positivity of Π_0 ensures that

$$q_1 q_2 \geq |\langle\psi_1|\psi_2\rangle|^2 = \cos^2(\theta)$$

The probability of an error result is

$$Q = Tr\left[\Pi_0\rho\right] = Tr\left[\Pi_0\left(r_1|\psi_1\rangle\langle\psi_1| + r_2|\psi_2\rangle\langle\psi_2|\right)\right] = r_1\langle\psi_1|\Pi_0|\psi_1\rangle + r_2\langle\psi_2|\Pi_0|\psi_2\rangle = r_1q_1 + r_2q_2$$

An optimal POVM should minimize Q, so we take $q_1q_2 = |\langle\psi_1|\psi_2\rangle|^2$ and plugging into Q we get

$$Q = r_1q_1 + r_2\frac{\cos^2(\theta)}{q_1}$$

and therefore

$$0 = \frac{dQ}{dq_1} = r_1 - r_2\frac{\cos^2(\theta)}{q_1^2}$$

Solving for q_1 and repeating the process for q_2 we find that

$$q_1 = \sqrt{\frac{r_2}{r_1}}\cos(\theta) = \sqrt{\frac{r_2}{r_1}}\,|\langle\psi_1|\psi_2\rangle| \qquad q_2 = \sqrt{\frac{r_1}{r_2}}\cos(\theta) = \sqrt{\frac{r_1}{r_2}}\,|\langle\psi_1|\psi_2\rangle|$$

Any derivative test you'd care to apply demonstrates that this is a minimum and plugging into Q we find that

$$Q_{min} = r_1q_1 + r_2q_2 = r_1\sqrt{\frac{r_2}{r_1}}\cos(\theta) + r_2\sqrt{\frac{r_1}{r_2}}\cos(\theta) = 2\sqrt{r_1r_2}\cos(\theta)$$

So we have a prescription for constructing an optimal POVM to unambiguously measure

$$\rho = r_1|\psi_1\rangle\langle\psi_1| + r_2|\psi_2\rangle\langle\psi_2|$$

using $p_j = 1 - q_j$ and $\cos(\theta) = |\langle\psi_1|\psi_2\rangle|$,

$$\Pi_1 = \frac{1 - \sqrt{\frac{r_2}{r_1}}\cos(\theta)}{\sin^2(\theta)}|\psi_2^\perp\rangle\langle\psi_2^\perp| \qquad \Pi_2 = \frac{1 - \sqrt{\frac{r_1}{r_2}}\cos(\theta)}{\sin^2(\theta)}|\psi_1^\perp\rangle\langle\psi_1^\perp| \qquad \Pi_0 = I - \Pi_1 - \Pi_2$$

This is not always optimal, since strange things happen when $\sin(\theta) = |\langle\psi_1^\perp|\psi_2\rangle| \approx 0$ or $r_j \approx 0$, but that's the sort of thing that homework is for.

Exercises

1) Error positive.

From the section on unambiguous state discrimination the declaration was made that in order for Π_0 to be a positive operator (in order for the POVM to exist) $q_1 q_2 \geq |\langle\psi_1|\psi_2\rangle|^2$, where q_1, q_2 are the failure probabilities for detection the corresponding states. Here you'll figure out where this equation comes from.

Although it isn't necessary, it may help to use $\cos(\theta) = |\langle\psi_1|\psi_2\rangle|$ and $\sin(\theta) = |\langle\psi_1|\psi_2^\perp\rangle| = |\langle\psi_1^\perp|\psi_2\rangle|$.

a) Write $\Pi_1 = \frac{p_1}{\sin^2(\theta)}|\psi_2^\perp\rangle\langle\psi_2^\perp|$, $\Pi_2 = \frac{p_2}{\sin^2(\theta)}|\psi_1^\perp\rangle\langle\psi_1^\perp|$, and then $\Pi_0 = I - \Pi_1 - \Pi_2$ as a 2×2 matrix in the $\{|\psi_1\rangle, |\psi_1^\perp\rangle\}$ basis.

b) Find the eigenvalues of Π_0.

c) Derive the condition for Π_0 to be a positive operator.

2) Fancier isn't always better.

Sometimes when trying to distinguish states in the ensemble $\{p(|\psi_1\rangle) = r_1, p(|\psi_2\rangle) = r_2\}$ we get a better success rate using just projective measurements. Define the measurement operators 1 and 2,

$$P_1 = |\psi_1\rangle\langle\psi_1| \qquad P_2 = |\psi_1^\perp\rangle\langle\psi_1^\perp|$$

a) Find the probability of seeing each result, given $|\psi_1\rangle$ then given $|\psi_2\rangle$.

b) One of the two results is unambiguous since it only occurs for one state. The other result is a "failure to discriminate" since it can occur for either input. Which operators and states are which?

c) What is the total failure probability, where "failure" means not being able to say with certainty which state was measured?

d) In the lecture we saw that the optimal (smallest) failure probabilities for $|\psi_1\rangle$ and $|\psi_2\rangle$ are

$$q_1 = \sqrt{\frac{r_2}{r_1}}\,|\langle\psi_1|\psi_2\rangle| \qquad q_2 = \sqrt{\frac{r_1}{r_2}}\,|\langle\psi_1|\psi_2\rangle|$$

q_1, q_2 are probabilities and therefore $0 \le q_1, q_2 \le 1$. For what values of r_1 does the optimal POVM, $\{\Pi_j\}$, from the lecture exist? When should you switch to using the projection measurement, $\{P_j\}$, from this exercise?

3) Tangle free.

In Neumark's theorem we perform a POVM measurement on space A by entangling it with some $|\phi\rangle \in B$ using U_{ab} and then doing a projective measurement on B in the $\{|m\rangle_b\}$ basis.

Assume that $U_{ab} = U_a \otimes U_b$ (it can be expressed as a pair of local operators) and that the detection operators are defined in the usual way, $A_m|\psi\rangle_a \equiv \langle m|_b U_{ab}|\psi\rangle_a|\phi\rangle_b$.

a) Given the state $|\psi\rangle_a$, what is the probability of result m?

b) What is the effect of measurement m on $|\psi\rangle_a$?

c) How do your answers to parts a and b change if you change $|\psi\rangle_a$ to the state $|\eta\rangle_a$?

d) How do your answers to parts a and b change if you change $|\phi\rangle_b$ to the state $|\mu\rangle_b$?

e) Explain what's happening here. Describe what this POVM is doing to states in A and why.