QI Lecture 17

Quantum Noise

So far we've considered isolated quantum systems and interacting sets of isolated quantum systems. We'll call the collection of such isolated, controlled systems "**the principle system**". While we hope to control everything about the principle system, nothing is perfect. The principle system interacts, at least a little, with other systems that we can't control or access. We call these systems "**the environment**". For example, if a photon carrying a qubit physically escapes and, say, hits the wall of the lab, then it's interacting very strongly with the environment; we have no way of recovering that lost qubit.

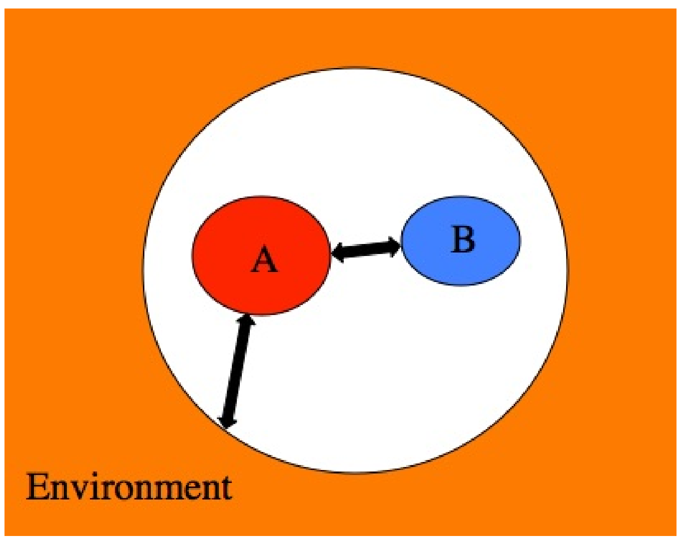

Figure 1: All of the interactions between parts of the Principle system, A and B, can be modeled with unitary operations and entanglement. Interactions with the environment are no different, but since we can't monitor or control the environment they manifest as undisclosed measurements and are modeled with "quantum operators".

"**Quantum operators**" map density matrices to density matrices.

$$\rho \longrightarrow \mathcal{E}[\rho]$$

We've seen before that we can describe undisclosed measurements, with detection operators $\{A_m\}$, acting on the density matrix ρ as

$$\rho' = \sum_m A_m \rho A_m^\dagger$$

Because this is a POVM, Neumark's theorem says it's equivalent to entangling the state $\rho \in A$ to another space, B, and then making an undisclosed measurement in B. That's an ideal way to model interactions with the environment; those interactions (like all interactions) are unitary, and since we can never know the state of the environment,[1] we trace over it.

In the context of quantum operations, we write

$$\mathcal{E}[\rho] = \sum_m E_m \rho E_m^\dagger$$

where E_m are the "**operation elements**". This is the "**Operator Sum Representation**" for a quantum operation. The simplest quantum operators are unitary transformations:

$$\mathcal{E}[\rho] = \sum_m E_m \rho E_m^\dagger = U \rho U^\dagger$$

The operator sum representation is more generalized than POVMs, because while a POVM maps states to states in the same space, quantum operations don't necessarily map to the same space. Not even the dimension of the space needs to stay the same for quantum operations.

Example The trace and partial trace are both quantum operations. Define the target space to be one dimensional, $B \equiv Span\{|0\rangle_b\}$, and define the operation elements to be

$$E_j = |0\rangle_b \langle j|_a$$

then it follows that

$$\mathcal{E}[\rho] = \sum_j E_j \rho E_j^\dagger = \sum_j |0\rangle_b \langle j|_a \rho |j\rangle_a \langle 0|_b = |0\rangle_b \left[\sum_j \langle j|_a \rho |j\rangle_a \right] \langle 0|_b = Tr[\rho] \, |0\rangle_b \langle 0|_b$$

We can do something similar for the partial trace. Define $E_k : A \otimes B \longrightarrow A$ as

[1] Some states in the environment we can be aware of, for example, the street address of your quantum lab. And when a photon is lost you know that it's still somewhere nearby (it's on its forward light cone until it runs into something). However, the polarization state (for example) of a single escaped photon is lost and unknowable in every useful sense.

$$E_k\left[\sum_j \alpha_j |v_j\rangle_a |j\rangle_b\right] \equiv \alpha_k |v_k\rangle_a$$

where $|v_j\rangle_a \in A$ are arbitrary states and $\mathcal{E}[\rho] = \sum_k E_k \rho E_k^\dagger$. We can write any density matrix in $A \otimes B$ as

$$\rho = \sum_{jk} C_{jk} \otimes |j\rangle_b \langle k|_b$$

where $C_{jk} \in A$. It follows that

$$\mathcal{E}[\rho] = \sum_k E_k \left(\sum_{ij} C_{ij} \otimes |i\rangle_b \langle j|_b\right) E_k^\dagger = \sum_k C_{kk} = Tr_b\left[\sum_{jk} C_{jk} \otimes |j\rangle_b \langle k|_b\right] = Tr_b\left[\rho\right]$$

In other words, $\mathcal{E}[\rho]$ and $Tr_b[\rho]$ are equal because they do the same thing.

■

The operator sum representation isn't unique. In fact, there's a simple relationship between indistinguishable quantum operations.

Theorem (Unitary Freedom in the Operator Sum Representation)**.** *Suppose* $\{E_1, E_2, \ldots, E_m\}$ *and* $\{F_1, F_2, \ldots, F_n\}$ *are the operation elements for quantum operations* $\mathcal{E}$ *and* $\mathcal{F}$*. Assuming, without loss of generality, that* $m \geq n$*, we can "pad"* $\{F_1, F_2, \ldots, F_n, 0, 0 \ldots\}$ *so that the two sets have the same number of operations.*

$\mathcal{E} = \mathcal{F}$ *if and only if* $\exists u_{jk} \in \mathbb{C}$ *such that* $E_j = \sum_k u_{jk} F_k$ *and* u_{jk} *are the elements of an* $m \times m$ *matrix.*

There are three important axioms that we'd like from quantum operations and given those axioms the operator sum representation is the only option.

Theorem (The Operator Sum Representation is the "Correct" Representation)**.** *Defining* $\mathcal{E}$ *as a map that acts on density matrices we can declare the following axioms:*

- *A1)* $Tr\left[\mathcal{E}[\rho]\right]$ *is the probability that the process represented by* $\mathcal{E}$ *occurs for initial state* ρ *and therefore* $0 \leq Tr\left[\rho\right] \leq 1$*.*
- *A2) For probabilities* $\{p_j\}$*, we have that* $\mathcal{E}\left[\sum_j p_j \rho_j\right] = \sum_j p_j \mathcal{E}\left[\rho_j\right]$*. This means that* $\mathcal{E}$ *is a "convex-linear map".*
- *A3)* $\mathcal{E}$ *is "completely positive", meaning that* $\mathcal{E}[A]$ *is a positive operator for any positive operator* A *(not just density operators) and that even if we extend our attention to a operators on a larger space,* $\left(\mathcal{E}_a \otimes I_b\right)\left[A\right]$ *is still a positive operator.*

The map $\mathcal{E}$ satisfies all three of these axioms if and only if

$$\mathcal{E}[\rho] = \sum_k E_k \rho E_k^\dagger$$

for some set of operators, $\{E_k\}$, such that $\sum_k E_k^\dagger E_k \leq I$.

When $Tr[\mathcal{E}[\rho]] = 1$ we call $\mathcal{E}$ "trace preserving". The probability of this happening is 1, typically because we're applying the channel on purpose or just assuming the channel is being applied (in the case of error) and this is the situation we'll focus on almost exclusively.

When $Tr[\mathcal{E}[\rho]] < 1$ we say $\mathcal{E}$ is "non-trace preserving". This corresponds to situations where classical information has been extracted and is being used to influence the behavior of the channel, such as in teleportation, entanglement distillation, or (as we'll see in the next lecture) error correction. We see this in action in teleportation (see lecture 12) because after Alice performs a Bell-basis measurement she reports the results to Bob who then executes one of four different procedures on the remaining qubit based on those results. The channel is changed based on some extracted classical information, and the probability of each of the possible channels happening is less than 1.

Your Intuition

when you see operations like this

$$\rho \quad \longrightarrow \quad \mathcal{E}[\rho] = \sum_m E_m \rho E_m^\dagger$$

should be to think about this in terms of an unrevealed measurement,

$$\rho \quad \longrightarrow \quad \rho' = \sum_m P_m \rho P_m^\dagger = \sum_m p(m) \rho_m$$

For example, if Alice and Bob share $|\Phi_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ and Bob measures his qubit in the computational basis and shares the result with Alice, then the state of her qubit will be, with probability $\frac{1}{2}$ each,

$$\frac{\langle 0|_b |\Phi_+\rangle}{\sqrt{\frac{1}{2}}} = \sqrt{2}\langle 0|_b \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = |0\rangle_a \qquad \frac{\langle 1|_b |\Phi_+\rangle}{\sqrt{\frac{1}{2}}} = \sqrt{2}\langle 1|_b \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = |1\rangle_a$$

On the other hand, if he hides the result but announces that he made the measurement, then all that Alice can say is "With probability $\frac{1}{2}$ each, my qubit is in the state $|0\rangle$ or $|1\rangle$." That ensemble is described by

$$\rho_a = p(0)P_0|\Phi_+\rangle\langle\Phi_+|P_0^\dagger + p(1)P_1|\Phi_+\rangle\langle\Phi_+|P_1^\dagger = \frac{1}{2}|0\rangle_a\langle 0|_a + \frac{1}{2}|1\rangle_a\langle 1|_a \sim \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

A summation of ket-bras is what it looks like when something is in *some* quantum state, but we don't know which. If Bob finally tells Alice the result of his measurement, then

$$\rho_a = \frac{1}{2}|0\rangle_a\langle 0|_a + \frac{1}{2}|1\rangle_a\langle 1|_a \quad \longrightarrow \quad \begin{cases} p(0) = \frac{1}{2} & , \frac{P_0\rho_a P_0}{p(0)} = \frac{|0\rangle_a\langle_a 0|\rho_a|0\rangle_a\langle_a 0|}{\frac{1}{2}} = |0\rangle_a\langle 0|_a \\ \\ p(1) = \frac{1}{2} & , \frac{P_1\rho_a P_1}{p(1)} = \frac{|1\rangle_a\langle_a 1|\rho_a|1\rangle_a\langle_a 1|}{\frac{1}{2}} = |1\rangle_a\langle 1|_a \end{cases}$$

In other words,

$$\boxed{\text{The summations in density matrices represent a lack of classical knowledge.}}$$

On yet another hand, what if Alice and Bob don't talk at all? Remember that entangled systems aren't magically connected to one another, so nothing that Bob does to his qubit has any direct impact on it. Whether or not Bob does a measurement, when we don't have access to a quantum system we trace over it, so Alice's density matrix is

$$\rho_a = Tr_b\left[|\Phi_+\rangle\langle\Phi_+|\right] = \langle 0|_b|\Phi_+\rangle\langle\Phi_+|0\rangle_b + \langle 1|_b|\Phi_+\rangle\langle\Phi_+|1\rangle_b = \frac{1}{2}|0\rangle_a\langle 0|_a + \frac{1}{2}|1\rangle_a\langle 1|_a$$

In other words,

$$\boxed{\text{The summations in density matrices represent having incomplete access to an entangled system.}}$$

So here, when we talk about quantum operations using the operator sum representation we're considering a very wide variety of channels. Unitary operators,

$$\rho \longrightarrow U\rho U^\dagger$$

represent isolated quantum systems, while operator sums,

$$\rho \longrightarrow \sum_m E_m\rho E_m^\dagger$$

represent systems that are being entangled with the environment; some different, inaccessible system.

$\boxed{\text{The behavior of entangled states is classical, when you have incomplete access to the state.}}$

You should always treat the separate terms in the operator sum representation as being distinguishable events that someone might know about. When Alice and Bob share $|\Phi_+\rangle$, Alice's state is $\rho_a = \frac{1}{2}|0\rangle_a\langle 0|_a + \frac{1}{2}|1\rangle_a\langle 1|_a$, but if Bob reports a measurement resulting in $|1\rangle_b$, then Alice's state updates, like a classical probability does, to $\rho_a = |1\rangle_a\langle 1|_a$.

The Bloch Vector

The density matrix for a qubit is a 2×2 hermitian matrix with trace 1. Any such matrix can be written

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2} = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + r_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + r_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$

where $\vec{r}$ is the "**Bloch vector**" for ρ.

In the Bloch sphere representation we can write a pure-state qubit as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

meaning that its density matrix is

$$\begin{aligned}
\rho &= |\psi\rangle\langle\psi| \\
&= \begin{bmatrix} \cos^2\left(\frac{\theta}{2}\right) & e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) & \sin^2\left(\frac{\theta}{2}\right) \end{bmatrix} \\
&= \tfrac{1}{2}\begin{bmatrix} 1+\cos(\theta) & e^{-i\phi}\sin(\theta) \\ e^{i\phi}\sin(\theta) & 1-\cos(\theta) \end{bmatrix} \\
&= \tfrac{1}{2}\begin{bmatrix} 1+\cos(\theta) & \cos(\phi)\sin(\theta) - i\sin(\phi)\sin(\theta) \\ \cos(\phi)\sin(\theta) + i\sin(\phi)\sin(\theta) & 1-\cos(\theta) \end{bmatrix} \\
&= \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \cos(\phi)\sin(\theta)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \sin(\phi)\sin(\theta)\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \cos(\theta)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \tfrac{I+\vec{r}\cdot\vec{\sigma}}{2}
\end{aligned}$$

where $\vec{r} = (\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta))$. Notice that

$$\|\vec{r}\| = \sqrt{\cos^2(\phi)\sin^2(\theta) + \sin^2(\phi)\sin^2(\theta) + \cos^2(\theta)} = \sqrt{\sin^2(\theta) + \cos^2(\theta)} = 1$$

Therefore, the Bloch vector for a pure state has magnitude 1. On the other hand, for a completely mixed state

$$\rho = \frac{1}{2}I \quad \Rightarrow \quad \|\vec{r}\| = 0$$

The Bloch vector is a decent measure of the purity of a qubit.

We can use the Bloch vector to see how different kinds of noise affect qubits.

Bit, Phase, and Bit-Phase Flip Channel[2]

These model basic errors that, individually or collectively, model situation-specific errors.

Bit Flip Channel In the bit flip channel, the qubit is left unchanged with probability $1 - p$ and is flipped, $|0\rangle \leftrightarrow |1\rangle$, with probability p.

$$E_0 = \sqrt{1-p}I = \sqrt{1-p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad\qquad E_1 = \sqrt{p}X = \sqrt{p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Writing $\rho = \frac{I + r_x X + r_y Y + r_z Z}{2}$ and using $\sigma_j \sigma_k = \delta_{jk} I + i\epsilon_{jk\ell}\sigma_\ell$, we can see the affect of the Bit Flip channel on Bloch vectors:

$$\begin{aligned}
\mathcal{E}[\rho] &= E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger \\
&= (1-p)\rho + pX\rho X \\
&= \tfrac{1-p}{2}(I + r_x X + r_y Y + r_z Z) + \tfrac{p}{2}X(I + r_x X + r_y Y + r_z Z)X \\
&= \tfrac{1-p}{2}(I + r_x X + r_y Y + r_z Z) + \tfrac{p}{2}\left(X^2 + r_x X^3 + r_y XYX + r_z XZX\right) \\
&= \tfrac{1-p}{2}(I + r_x X + r_y Y + r_z Z) + \tfrac{p}{2}\left(I + r_x X + r_y iZX + r_z iXY\right) \\
&= \tfrac{1-p}{2}(I + r_x X + r_y Y + r_z Z) + \tfrac{p}{2}\left(I + r_x X - r_y Y - r_z Z\right) \\
&= \tfrac{1}{2}I + \tfrac{1}{2}r_x X + \left(\tfrac{1}{2} - p\right) r_y Y + \left(\tfrac{1}{2} - p\right) r_z Z
\end{aligned}$$

To see what that means, consider what $\vec{r} = (1, 0, 0)$ means:

$$\rho = \frac{1}{2}(I + X) \sim \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \sim |+\rangle\langle +|$$

$|+\rangle$, the 1-eigenstate of X, is the x-axis of the Bloch sphere. In other words, the $\hat{y}$ and $\hat{z}$ directions contract while the $\hat{x}$ direction stays the same.

[2] Say that five times fast.

If there's a $p = \frac{1}{2}$ probability of a bit flip, then there's no information whatsoever about what the bit is, and the Y and Z components of the Bloch vector are destroyed.

■

Phase Flip Channel

$$E_0 = \sqrt{1-p}I = \sqrt{1-p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_1 = \sqrt{p}Z = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} E_0\rho E_0^\dagger + E_1\rho E_1^\dagger &= (1-p)\rho + pZ\rho Z \\ &= \tfrac{1}{2}I + \left(\tfrac{1}{2}-p\right)r_x X + \left(\tfrac{1}{2}-p\right)r_y Y + \tfrac{1}{2}r_z Z \end{aligned}$$

$$\vec{r} = (0,0,1) \quad \Rightarrow \quad \rho = \frac{1}{2}(I+Z) \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim |0\rangle\langle 0|$$

$|0\rangle$ is the 1-eigenstate of Z and the z axis of the Bloch sphere. Once again, $\hat{z}$ is left alone while the other two directions, $\hat{x}$ and $\hat{y}$, contract.

■

Bit-Phase Flip Channel

$$E_0 = \sqrt{1-p}I = \sqrt{1-p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_1 = \sqrt{p}Y = \sqrt{p}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\begin{aligned} E_0\rho E_0^\dagger + E_1\rho E_1^\dagger &= (1-p)\rho + pY\rho Y \\ &= \tfrac{1}{2}I + \left(\tfrac{1}{2}-p\right)r_x X + \tfrac{1}{2}r_y Y + \left(\tfrac{1}{2}-p\right)r_z Z \end{aligned}$$

$$\vec{r} = (0,1,0) \quad \Rightarrow \quad \rho = \frac{1}{2}(I+Y) \sim \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2}i & \frac{1}{2} \end{bmatrix} \sim \left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0| - i\langle 1|}{\sqrt{2}}\right)$$

$\frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ is the 1-eigenstate of Y and the y axis of the Bloch sphere. As you may have been expecting, $\hat{y}$ is left alone while the other two directions, $\hat{x}$ and $\hat{z}$, contract.

■

Depolarization Channel

$$\boxed{\mathcal{E}[\rho] = (1-p)\rho + p\frac{1}{2}I}$$

The depolarization channel describes a very simple situation: with probability $1-p$ nothing happens to the qubit, and with probability p all of the information is lost. This is represented by the "completely mixed state"

$$\frac{1}{2}I \sim \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

which is two-dimensional because the system in question is a qubit.

The depolarizing channel is destructive in a very straightforward way; with probability p, ρ is replaced with the completely mixed state, $\frac{1}{2}I$. For $p=1$ the qubit is completely scrambled, so $\vec{r} \to 0$, and for $p=0$ the qubit is left unharmed. Regardless of the original ρ the depolarization channel shrinks the Bloch vector, which means that unlike the earlier channels this one shrinks the Bloch sphere in every direction.

Although the form above, $\mathcal{E}[\rho] = (1-p)\rho + p\frac{1}{2}I$, is easier to read, it isn't written in terms of operation elements. To change that, we first recognize[3] that for any ρ representing a qubit, we have that $I = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{2}$, which implies that the operator sum representation is

$$\begin{aligned}
\mathcal{E}[\rho] &= (1-p)\rho + p\tfrac{1}{2}I \\
&= (1-p)\rho + p\left(\tfrac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}\right) \\
&= \left(1 - \tfrac{3p}{4}\right)\rho + \tfrac{p}{4}X\rho X + \tfrac{p}{4}Y\rho Y + \tfrac{p}{4}Z\rho Z \\
&= \textstyle\sum_{j=0}^{3} E_j \rho E_j^\dagger
\end{aligned}$$

where

$$E_j \equiv \begin{cases} \sqrt{1-\frac{3p}{4}}I & ,j=0 \\ \frac{\sqrt{p}}{2}\sigma_j & ,j\neq 0 \end{cases}$$

d-Dimensional States

$$\mathcal{E}[\rho] = p\frac{1}{d}I + (1-p)\rho$$

[3] This isn't obvious, which is why you'll do some homework.

is the simple generalization of the Depolarizing Channel for d-dimensional state spaces. With probability p the state is scrambled/lost, which we represent by replacing it with a completely mixed state, $\frac{1}{d}I$.

Amplitude Damping Channel

One of the states of the qubit, typically $|0\rangle$, is indistinguishable from an error. This is a very reasonable kind of error to encounter when using a "**Fock**[4] **Space**", where $|j\rangle$ represents the state of some channel having j photons in it. Typically the places where photons can be found are called "modes".

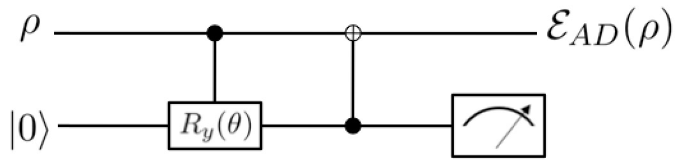

Figure 2: The circuit for amplitude damping. There is some probability that a $|1\rangle$ in the principle mode will give rise to a $|1\rangle$ in the environmental mode, determined by θ. The total number of "photons" should be conserved by this operation, so if the state of the environment is $|1\rangle$, then the CNOT ensures that the state of the principle mode is $|0\rangle$.

Being concerned for the moment with qubits, we look at states of the form $\alpha|0\rangle + \beta|1\rangle$. The loss of a photon can be modeled by coupling the mode to another mode with a beam splitter, S, and then tracing out over the new mode. If the second mode starts out empty, then

$$S\left[\alpha|0\rangle_a + \beta|1\rangle_a\right]|0\rangle_b = \alpha|0\rangle_a|0\rangle_b + \beta\cos(\theta)|1\rangle_a|0\rangle_b + \beta\sin(\theta)|0\rangle_a|1\rangle_b$$

where θ is just a parameter to describe the beam splitter; when $\theta = 0$ all the light passes through and when $\theta = \frac{\pi}{2}$ all of the light gets redirected. "Amplitude Damping" is so important, that its quantum operation gets its own subscript, $\mathcal{E}_{AD}$. If ρ is the density matrix of the qubit, then the operator sum representation is

$$\mathcal{E}_{AD}[\rho] = \langle 0|_b S\left(\rho\otimes|0\rangle_b\langle 0|_b\right)S^\dagger|0\rangle_b + \langle 1|_b S\left(\rho\otimes|0\rangle_b\langle 0|_b\right)S^\dagger|1\rangle_b = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$$

where

$$E_j = \langle j|_b S|0\rangle_b$$

[4]Careful here. That's "Fock" as in "lock" not "luck".

Although it is not at all obvious from the notation, the operation elements are maps, not scalars. $E_j : A \to A$ because while $S : A \otimes B \to A \otimes B$, $|0\rangle_b$ "blocks the input" and $\langle j|_b$ "blocks the output" from B.

We find that

$$E_0|0\rangle_a = \langle 0|_b S|0\rangle_a|0\rangle_b = |0\rangle_a \qquad E_0|1\rangle_a = \langle 0|_b S|1\rangle_a|0\rangle_b = \cos(\theta)|1\rangle_a$$

$$E_1|0\rangle_a = \langle 0|_b S|0\rangle_a|0\rangle_b = 0 \qquad E_1|1\rangle_a = \langle 0|_b S|1\rangle_a|0\rangle_b = \sin(\theta)|1\rangle_a$$

and therefore

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \cos(\theta) \end{bmatrix} \qquad E_1 = \begin{bmatrix} 0 & \sin(\theta) \\ 0 & 0 \end{bmatrix}$$

Clearly the effect of the amplitude damping channel, especially after many applications, is to pull input states closer to $|0\rangle$, and in fact that's exactly what happens. Writing qubits using the Bloch vector representation, $\rho = \frac{I+\vec{r}\cdot\vec{\sigma}}{2}$, we find[5] that first after one and then after many applications

$$\vec{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \longrightarrow \begin{pmatrix} r_x\cos(\theta) \\ r_y\cos(\theta) \\ r_z\cos^2(\theta) + \sin^2(\theta) \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Which should seem about right, since for $\vec{r} = (0,0,1)$

$$\rho = \frac{I+Z}{2} \sim \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim |0\rangle\langle 0|$$

Example In a cavity with a single quanta of energy and a single atom, we can describe where that energy is using space A for the atom and space B for the EM field of the cavity.

With zero detuning[6], the unitary evolution of the system after time t is given by

$$U(t) = |00\rangle\langle 00| + \cos(gt)\left[|01\rangle\langle 01| + |10\rangle\langle 10|\right] - i\sin(gt)\left[|01\rangle\langle 10| + |10\rangle\langle 01|\right]$$

If the cavity initially has no photons, $|0\rangle_b$, and the atom starts in a superposition of its ground and excited states, $\alpha|0\rangle_a+\beta|1\rangle_a$, then the system starts in the state $|\psi\rangle = \alpha|00\rangle+\beta|10\rangle$ with density matrix

$$\rho = \rho_a \otimes |0\rangle_b\langle 0|_b = \left[\alpha|0\rangle_a + \beta|1\rangle_a\right]\left[\alpha\langle 0|_a + \beta\langle 1|_a\right] \otimes |0\rangle_b\langle 0|_b$$

If we focus on the state of the atom and trace over the cavity, then we have a quantum operation.

[5] Or rather, *you'll* find. In the homework.

[6] The resonant frequency of the cavity and the absorption/emission frequency of the atom are equal.

$$\begin{aligned}
\mathcal{E}[\rho] &= Tr_b\left[U\rho U^\dagger\right] \\
&= \langle 0|_b U\rho U^\dagger|0\rangle_b + \langle 1|_b U\rho U^\dagger|1\rangle_b \\
&= \begin{cases} & \left(|0\rangle_a\langle 00| + \cos(gt)|1\rangle_a\langle 10| - i\sin(gt)|1\rangle_a\langle 01|\right)\rho\left(|00\rangle\langle 0|_a + \cos(gt)|10\rangle\langle 1|_a + i\sin(gt)|01\rangle\langle 1|_a\right) \\ + & \left(\cos(gt)|0\rangle_a\langle 01| - i\sin(gt)|0\rangle_a\langle 10|\right)\rho\left(\cos(gt)|01\rangle\langle 0|_a + i\sin(gt)|10\rangle\langle 0|_a\right) \end{cases} \\
&= \begin{cases} & \left(|0\rangle_a\langle 0|_a + \cos(gt)|1\rangle_a\langle 1|_a\right)\rho_a\left(|0\rangle_a\langle 0|_a + \cos(gt)|1\rangle_a\langle 1|_a\right) \\ + & \left(-i\sin(gt)|0\rangle_a\langle 1|_a\right)\rho_a\left(i\sin(gt)|1\rangle_a\langle 0|_a\right) \end{cases} \\
&= \begin{cases} & \left(|0\rangle_a\langle 0|_a + \cos(gt)|1\rangle_a\langle 1|_a\right)\rho_a\left(|0\rangle_a\langle 0|_a + \cos(gt)|1\rangle_a\langle 1|_a\right) \\ + & \left(\sin(gt)|0\rangle_a\langle 1|_a\right)\rho_a\left(\sin(gt)|1\rangle_a\langle 0|_a\right) \end{cases} \\
&= E_0\rho_a E_0^\dagger + E_1\rho_a E_1^\dagger
\end{aligned}$$

Where, in the $\{|0\rangle_a, |1\rangle_a\}$ basis,

$$E_0 = \langle 0|_b U(t)|0\rangle_b \sim \begin{bmatrix} 1 & 0 \\ 0 & \cos(gt) \end{bmatrix} \qquad E_1 = \langle 1|_b U(t)|0\rangle_b \sim \begin{bmatrix} 0 & \sin(gt) \\ 0 & 0 \end{bmatrix}$$

Evidently, the spontaneous emission of a photon (in this case into a cavity) is an example of amplitude damping for the atom.

■

Generalized Amplitude Damping

Notice that the amplitude damping channel is a little ideal; photons disappear as though vanishing into a sink with zero temperature. We can model a non-zero temperature by assuming that the second mode already has photons that occasionally escape into the principle mode; this is what makes this channel "Generalized", $\mathcal{E}_{GAD}$.[7]

The model for this is

$$E_0 = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & \cos(\theta) \end{bmatrix} \qquad E_1 = \sqrt{p}\begin{bmatrix} 0 & \sin(\theta) \\ 0 & 0 \end{bmatrix}$$

$$E_2 = \sqrt{1-p}\begin{bmatrix} \cos(\theta) & 0 \\ 0 & 1 \end{bmatrix} \qquad E_3 = \sqrt{1-p}\begin{bmatrix} 0 & 0 \\ \sin(\theta) & 0 \end{bmatrix}$$

[7] I can't help but suspect that somebody somewhere insisted on this notation because they think the word "egad" is funny.

With probability p, the regular AD channel is applied and a photon is removed from the principle mode, and with probability $1-p$ a photon is added. So here we're describing temperature using probabilities. The stationary, equilibrium density matrix is

$$\rho_\infty = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}$$

and satisfies $\mathcal{E}_{GAD}[\rho_\infty] = \rho_\infty$.

Phase Damping

There are many reasons that we can lose track of the relative phase between different states.

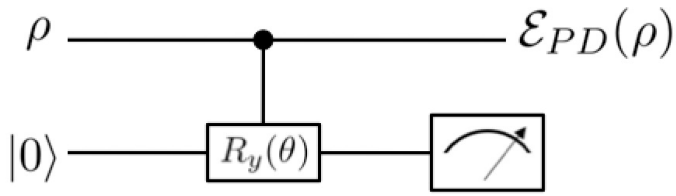

Figure 3: The circuit for phase damping. This circuit is somewhat more "gentle" than amplitude damping channel. It's simply a weak measurement of the principle system.

We model phase damping on a qubit,

$$\rho = |\psi\rangle = [\alpha|0\rangle + \beta|1\rangle][\alpha\langle 0| + \beta\langle 1|] \sim \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}$$

with a "phase kick", which is a rotation by $R_z(\theta)$, through a random angle θ. Because Gaussians are both commonly encountered and easy to work with, we describe the probability distribution of θ to be normally distributed with mean 0 and variance 2μ.

$$\mathcal{E}[\rho] = \frac{1}{\sqrt{4\pi\mu}} \int \left[R_z(\theta)|\psi\rangle\langle\psi|R_z^\dagger(\theta)\right] e^{-\frac{\theta^2}{4\mu}}\, d\theta \sim \begin{bmatrix} |\alpha|^2 & \alpha\beta^* e^{-\mu} \\ \alpha^*\beta e^{-\mu} & |\beta|^2 \end{bmatrix}$$

For the coherent state $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ there's an equal chance of measuring this and seeing either $|0\rangle$ or $|1\rangle$. The difference between $|+\rangle$ and the ensemble $\left\{p(|0\rangle) = \frac{1}{2}, p(|1\rangle) = \frac{1}{2}\right\}$ is coherence, which is measured through interference and clearly indicated by the density matrices

$$|+\rangle\langle +| \sim \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad\qquad \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

The phase damping channel strips away the off-diagonal terms (which correspond to coherence between the basis states) exponentially.

The operation elements for this channel are

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix} \qquad E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}$$

Remarkably, using $a = \sqrt{\frac{1+\sqrt{1-\lambda}}{2}}$ and $b = \sqrt{\frac{1-\sqrt{1-\lambda}}{2}}$, we have a unitary relationship

$$F_0 = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = aE_0 + bE_1 \qquad F_1 = b \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = bE_0 - aE_1$$

which means, according to the "Unitary Freedom in the Operator Sum Representation" theorem as well as the section on the phase flip, that the phase damping channel and the phase flip channels are <u>the same channel</u>.

Exercises

1) Depolarization Double Check

When finding a way to write the depolarization channel using operation elements, I flippantly declared that

$$I = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{2}$$

and here you'll prove it. Define

$$\mathcal{E}[A] = \frac{A + XAX + YAY + ZAZ}{4}$$

a) Show that

$$\mathcal{E}[\sigma_j] = 0$$

where $\sigma_j = X, Y, Z$.
(Hint: It may help to use the fact that $XY = iZ$, $YZ = iX$, and $ZX = iY$.)

b) Show that

$$\mathcal{E}[I] = I$$

c) Using $\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$, show that

$$I = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{2}$$

2) Boltzmann From The Blue

For the generalized amplitude damping channel, we can define the temperature, T, of a qubit in equilibrium using

$$p(0) = \frac{1}{\mathcal{Z}} e^{-\frac{E_0}{k_B T}} \qquad\qquad p(1) = \frac{1}{\mathcal{Z}} e^{-\frac{E_1}{k_B T}}$$

where k_B is Boltzmann's constant, E_j is the energy of the state $|j\rangle$, and

$$\mathcal{Z} = e^{-\frac{E_0}{k_B T}} + e^{-\frac{E_1}{k_B T}}$$

What temperature describes the state ρ_∞?

3) Phase

Using $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix}$, $E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}$, and $\rho = |\psi\rangle\langle\psi| \sim \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}$ calculate the effect of the phase damping channel, $\mathcal{E}[\rho]$, and write the result as a 2×2 density matrix.

Describe the effect in words.

4) Real Bad Amplitude

Using the operation elements from the amplitude damping channel

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \cos(\theta) \end{bmatrix} \qquad E_1 = \begin{bmatrix} 0 & \sin(\theta) \\ 0 & 0 \end{bmatrix}$$

show that the effect on the Bloch vector is

$$\vec{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \longrightarrow \begin{pmatrix} r_x \cos(\theta) \\ r_y \cos(\theta) \\ r_z \cos^2(\theta) + \sin^2(\theta) \end{pmatrix}$$