QI Lecture 18

Quantum Error Correction

Classical Bit-Flip Error Correction

Modern computers are paragons of perfect computation. An ordinary computer tends to flip bits at a rate of about 1 per 10^{17} operations. The best that we can expect from quantum computers today are error rates that hover around 1 error per 100-1000 operations. So error correction is a very serious concern for quantum computers.

Error correction is nothing new. The oldest method of error correction is redundancy.[1] If you send a single bit and there's a bit-flip error, then it disappears without a ripple. But if you send three bits, 000 or 111, and one of them gets flipped, then you've got two others to check against. This is called "**majority voting**" and these tripled bits are called "logical zero" and "logical one".

$$|0_L\rangle \equiv |000\rangle \qquad\qquad |1_L\rangle \equiv |111\rangle$$

If the probability of a single bit flipping is p, where $0 \leq p \leq \frac{1}{2}$, then the probability of a logical bit flipping (that is, of two or three out of three bits flipping) is $p^3 + 3p^2(1-p)$. Since for $0 < p < \frac{1}{2}$

$$p^3 + 3p^2(1-p) < p$$

repeating the same bit three times (not surprisingly) is an improvement. In fact, for very small error rates with $p << 1$, using logical bits of this kind drops the bit-flip probability to $\approx 3p^2$, which is extremely small for small p.

Three Qubit Bit-Flip Error Correction

Quantum information theory has a few things working against it that classical information doesn't. First and foremost, in classical information there's nothing to stop you from looking at the bits, comparing them, and fixing them (as with majority voting). Quantum error correct needs to be done "entirely in the dark" since a measured qubit is a bit.

[1] The oldest method of error correction is redundancy.

Quantum states can't be copied the way classical states can be;[2] the closest we can get is to entangled states like this

$$\alpha|0\rangle + \beta|1\rangle \quad \longrightarrow \quad \alpha|000\rangle + \beta|111\rangle$$

which we can do with a couple of CNOT gates

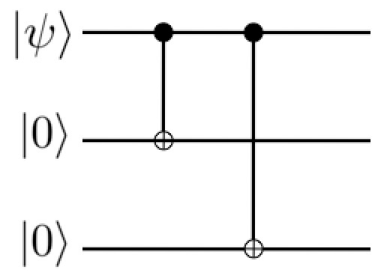

Figure 1: A simple circuit to entangle (not copy) states, $[\alpha|0\rangle + \beta|1\rangle]\,|0\rangle|0\rangle \longrightarrow \alpha|000\rangle + \beta|111\rangle$.

Suppose that all three of these qubits passes through a separate bit-flip channel,

$$\mathcal{E}(\rho) = (1-p)\rho + pX\rho X$$

As long as only one of the channels flips a bit, we can still recover the original state. There are four "**error syndromes**" corresponding to the results of a projective measurement $\{P_j\}$ with projection operations

$$\boxed{\begin{aligned} P_0 &= |000\rangle\langle 000| + |111\rangle\langle 111| \\ P_1 &= |100\rangle\langle 100| + |011\rangle\langle 011| \\ P_2 &= |010\rangle\langle 010| + |101\rangle\langle 101| \\ P_3 &= |001\rangle\langle 001| + |110\rangle\langle 110| \end{aligned}}$$

which allow us to determine which qubit was flipped (1, 2, or 3) or if none were flipped (0). More importantly, these projective measurements don't "damage" the states they measure; the state is left the same (assuming that the only error was a single bit flip).

For each of the four error syndromes there's a simple fix: simply apply X to the afflicted bit.

[2]This is the "no cloning theorem" and we'll get into it later.

Example If the third qubit is flipped, then

$$|\psi\rangle = \alpha|000\rangle + \beta|111\rangle \quad \longrightarrow \quad |\psi^{(3)}\rangle = \alpha|001\rangle + \beta|110\rangle$$

and therefore the probabilities for each error syndrome are

$$\begin{aligned} p(0) &= \langle\psi^{(3)}|P_0|\psi^{(3)}\rangle &= 0 \\ p(1) &= \langle\psi^{(3)}|P_1|\psi^{(3)}\rangle &= 0 \\ p(2) &= \langle\psi^{(3)}|P_2|\psi^{(3)}\rangle &= 0 \\ p(3) &= \langle\psi^{(3)}|P_3|\psi^{(3)}\rangle &= 1 \end{aligned}$$

and so the effect on the state of the (guaranteed) result is

$$\frac{P_3|\psi^{(3)}\rangle}{\sqrt{p(3)}} = \frac{[|001\rangle\langle 001| + |110\rangle\langle 110|]\,[\alpha|001\rangle + \beta|110\rangle]}{\sqrt{1}} = \alpha|001\rangle + \beta|110\rangle = |\psi^{(3)}\rangle$$

The amplitudes, α and β, are left unchanged by this measurement!

In this example, the error syndrome is 3, so we can apply X to the third qubit and recover the original state:

$$(I \otimes I \otimes X)\,[\alpha|001\rangle + \beta|110\rangle] = \alpha|000\rangle + \beta|111\rangle$$

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An alternative measurement for determining the syndrome is to use the observables Z_1Z_2 and Z_2Z_3. This is because

$$Z_1Z_2 = \left[|0\rangle_1\langle 0|_1 - |1\rangle_1\langle 1|_1\right]\left[|0\rangle_2\langle 0|_2 - |1\rangle_2\langle 1|_2\right] = \left[|00\rangle\langle 00| + |11\rangle\langle 11|\right] - \left[|01\rangle\langle 01| + |10\rangle\langle 10|\right]$$

meaning that if the qubits are the same, then we'll see a "1" result, and if we see a "-1", then they must be different. By looking at the results of both Z_1Z_2 and Z_2Z_3 we can determine which qubit has been flipped. If the first qubit is flipped, then 1 and 2 will be different while 2 and 3 will be the same. If the second qubit is flipped, the 2 will be different from both 1 and 3.

The error syndromes (the qubit that has been flipped) given the results of these measurements are:

Syndrome	Z_1Z_2	Z_2Z_3
0	1	1
1	-1	1
2	-1	-1
3	1	-1

Three Qubit Phase-Flip Error Correction

In a phase-flip error, the relative phase between $|0\rangle$ and $|1\rangle$ is flipped. Since global phase is unimportant, we can model this with a Z operation, since $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$. Since there's no classical analog to phase, phase-flip error is a strictly quantum phenomenon. Despite that, there's an easy way to turn the phase-flip channel into the bit-flip channel, allowing us to apply a nearly identical solution. The tripled state $\alpha|000\rangle + \beta|111\rangle$ doesn't help us to fix the phase-flip error, in part because there's no way to tell which qubit was phase flipped

$$Z_1 \otimes I_2 \otimes I_3|\psi\rangle = I_1 \otimes Z_2 \otimes I_3|\psi\rangle = I_1 \otimes I_2 \otimes Z_3|\psi\rangle = \alpha|000\rangle - \beta|111\rangle$$

This is ultimately because $|0\rangle$ and $|1\rangle$ are the eigenstates of Z. So, in order to deal with phase-flips we can change the basis of our qubits back and forth between $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$ and using Hadamard gates.

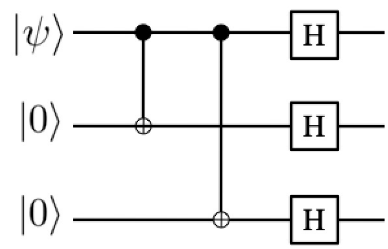

Figure 2: A circuit to transform $[\alpha|0\rangle + \beta|1\rangle]\,|0\rangle|0\rangle \longrightarrow \alpha|+++\rangle + \beta|---\rangle$.

Since $Z : |+\rangle \leftrightarrow |-\rangle$, we have essentially changed the phase-flip channel into a bit-flip channel. For this state the effect of the phase-flip channel

$$\mathcal{E}(\rho) = (1-p)\rho + pZ\rho Z$$

is

$$\alpha|+++\rangle + \beta|---\rangle \longrightarrow \begin{cases} \alpha|-++\rangle + \beta|+--\rangle \\ \alpha|+-+\rangle + \beta|-+-\rangle \\ \alpha|++-\rangle + \beta|--+\rangle \end{cases}$$

depending on which qubit is affected. Clearly, this is (in effect) the same as the bit-flip channel. The measurements we use to produce the error syndromes reflect that

$$\begin{aligned} P_0' &= HP_0H &= |+++\rangle\langle+++| + |---\rangle\langle---| \\ P_1' &= HP_1H &= |-++\rangle\langle-++| + |+--\rangle\langle+--| \\ P_2' &= HP_2H &= |+-+\rangle\langle+-+| + |-+-\rangle\langle-+-| \\ P_3' &= HP_3H &= |++-\rangle\langle++-| + |--+\rangle\langle--+| \end{aligned}$$

Since $Z = HXH$ and $X = HZH$, Hadamard gates also switch back and forth between X and Z.

$$\begin{aligned} 0 &\to I_1 \otimes I_2 \otimes I_3 & & \\ 1 &\to Z_1 \otimes I_2 \otimes I_3 &=& \; H_1X_1H_1 \otimes I_2 \otimes I_3 \\ 2 &\to I_1 \otimes Z_2 \otimes I_3 &=& \; I_1 \otimes H_2X_2H_2 \otimes I_3 \\ 3 &\to I_1 \otimes I_2 \otimes Z_3 &=& \; I_1 \otimes I_2 \otimes H_3X_3H_3 \end{aligned}$$

In keeping with the parallels between the bit-flip and phase-flip codes, an alternative measurement for determining the syndrome is to use the observables X_1X_2 and X_2X_3. This works because

$$X_1X_2 = [|+\rangle_1\langle+|_1 - |-\rangle_1\langle-|_1]\,[|+\rangle_2\langle+|_2 - |-\rangle_2\langle-|_2] = [|++\rangle\langle++|+|--\rangle\langle--|] - [|+-\rangle\langle+-|+|-+\rangle\langle-+|]$$

By looking at the results of both X_1X_2 and X_2X_3 we can determine which phase has been flipped. In what is essentially an exact repeat of the bit-flip error syndrome:

Syndrome	X_1X_2	X_2X_3
0	1	1
1	−1	1
2	−1	−1
3	1	−1

Example If the second qubit is phase-flipped, then

$$|\psi\rangle = \alpha|+++\rangle + \beta|---\rangle \quad\longrightarrow\quad |\psi^{(2)}\rangle = \alpha|+-+\rangle + \beta|-+-\rangle$$

Applying the observable X_1X_2 we find that

$$X_1X_2|\psi^{(2)}\rangle = \alpha X_1X_2|+-+\rangle + \beta X_1X_2|-+-\rangle = (1)(-1)\alpha|+-+\rangle + (-1)(1)\beta|-+-\rangle = -|\psi^{(2)}\rangle$$

meaning that $|\psi^{(2)}\rangle$ is a -1 eigenstate of X_1X_2 and so the result of the observation must be -1. Taking a step back from the "observable formalism", the fact that we can write the spectral decomposition of X_1X_2 as

$$X_1X_2 = (+1)|++\rangle\langle++| + (+1)|--\rangle\langle--| + (-1)|+-\rangle\langle+-| + (-1)|-+\rangle\langle-+|$$

means that the observable X_1X_2 is just a short-hand way of talking about the projective measurement with projective operators $P_1 = |++\rangle\langle++| + |--\rangle\langle--|$ and $P_{-1} = |+-\rangle\langle+-| + |-+\rangle\langle-+|$.

Applying the observable X_2X_3 we find that

$$X_2X_3|\psi^{(2)}\rangle = \alpha X_2X_3|+-+\rangle + \beta X_2X_3|-+-\rangle = (-1)(1)\alpha|+-+\rangle + (1)(-1)\beta|-+-\rangle = -|\psi^{(2)}\rangle$$

meaning that $|\psi^{(2)}\rangle$ is a -1 eigenstate of X_2X_3 and so the result of the observation must be -1. We have determined that qubits 1 and 2 have opposite phases ($X_1X_2 \longrightarrow -1$)

and that qubits 2 and 3 have opposite phases ($X_2X_3 \longrightarrow -1$), meaning that qubit 2 must have been phase-flipped (assuming there was only one phase-flipping).

Because the error syndrome is 2, so we apply Z to the second qubit and recover the original state:

$$(I_1 \otimes Z_2 \otimes I_3)\left[\alpha|+-+\rangle + \beta|-+-\rangle\right] = \alpha|+++\rangle + \beta|---\rangle$$

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Shor Code

Remarkably, we can describe *any* arbitrary qubit error as a combination of bit-flip and phase-flip errors, and we can repair them using a combination of the two corrective procedures: the Shor code.

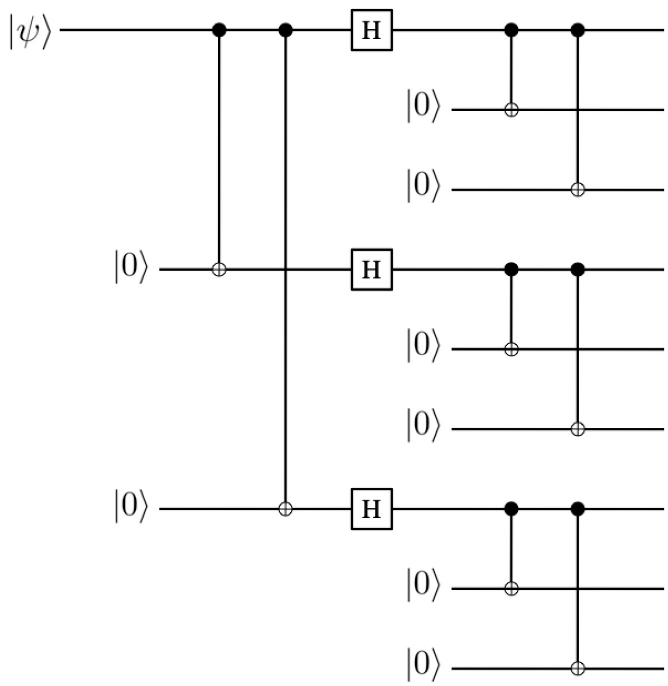

Figure 3: The circuit for creating the Shor Code's logical qubits.

This is created by first running the qubit through the circuit used for the phase-flip code

$$
\begin{aligned}
|0\rangle \quad &\longrightarrow \quad |+\rangle|+\rangle|+\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \\
|1\rangle \quad &\longrightarrow \quad |-\rangle|-\rangle|-\rangle = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)
\end{aligned}
$$

and then running all three of those qubits through the circuit used for the bit-flip code to yield the 9-qubit[3] Shor code

$$
\boxed{
\begin{aligned}
|0_L\rangle &\equiv \left(\frac{|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \\
|1_L\rangle &\equiv \left(\frac{|0\rangle|0\rangle|0\rangle - |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle - |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle - |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)
\end{aligned}
}
$$

The Shor Code encodes the state

$$
|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \longrightarrow \quad |\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle
$$

where $|\psi_L\rangle$ is non-standard, but not unreasonable, notation.

Shor Error Syndrome

The Shor code corrects both bit-flip and phase-flip errors on each of its nine qubits. To determine the error syndrome, we can look for bit-flips and phase-flips separately.

First, we'll check for bit flips by checking all three blocks, one at a time, in the same way that we did for the bit-flip code using

$$
\begin{aligned}
P_0 &= |000\rangle\langle 000| + |111\rangle\langle 111| \\
P_1 &= |100\rangle\langle 100| + |011\rangle\langle 011| \\
P_2 &= |010\rangle\langle 010| + |101\rangle\langle 101| \\
P_3 &= |001\rangle\langle 001| + |110\rangle\langle 110|
\end{aligned}
$$

or the observables Z_1Z_2 and Z_2Z_3 (followed by Z_4Z_5 and Z_5Z_6, then Z_7Z_8 and Z_8Z_9).

A phase-flip on any one of the qubits in a block flips the sign of the entire block. For example,

$$
Z_3\left[|0\rangle_1|0\rangle_2|0\rangle_3 + |1\rangle_1|1\rangle_2|1\rangle_3\right] = |0\rangle_1|0\rangle_2|0\rangle_3 + |1\rangle_1|1\rangle_2\left(-|1\rangle_3\right) = |0\rangle_1|0\rangle_2|0\rangle_3 - |1\rangle_1|1\rangle_2|1\rangle_3
$$

[3]There's also a 5 qubit Shor code

So when determining this second half of the error syndrome, the phase-flip, we can check entire blocks at a time. We can do that by comparing the same/different-ness using the observables $X_1X_2X_3X_4X_5X_6$ and then $X_4X_5X_6X_7X_8X_9$.

This works because, for example, if the phase between the first two blocks is different, then

$$\begin{aligned} & X_1X_2X_3X_4X_5X_6\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \\ = & \left(\frac{|1\rangle|1\rangle|1\rangle+|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right)\left(\frac{|1\rangle|1\rangle|1\rangle-|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right) \\ = & -\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \end{aligned}$$

and if they're the same

$$\begin{aligned} & X_1X_2X_3X_4X_5X_6\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \\ = & \left(\frac{|1\rangle|1\rangle|1\rangle-|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right)\left(\frac{|1\rangle|1\rangle|1\rangle-|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right) \\ = & (-1)(-1)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \\ = & \left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \end{aligned}$$

Recovery

To recover from a bit-flip error, we just flip the qubit back by applying the appropriate X_j.

To recover from a phase-flip error, we can apply $Z_1Z_2Z_3$, $Z_4Z_5Z_6$, or $Z_7Z_8Z_9$ to the appropriate block.

Example (fixing bit-flips) Sadly, the fourth qubit has been bit-flipped and the state of the logical qubit is

$$\alpha\,(\cdots)\left(\frac{|1\rangle|0\rangle|0\rangle+|0\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)+\beta\,(\cdots)\left(\frac{|1\rangle|0\rangle|0\rangle-|0\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)$$

and we have to find and correct the error.

First, we apply the bit-flip measuring projection operators, $\{P_0\}$, to each of the three blocks. On the first and third blocks the error syndrome is 0, but on the second the error syndrome is 1. Evidently, the first qubit in the second block (the fourth qubit overall) has been flipped.

Second, we check for phase flips using $X_1X_2X_3X_4X_5X_6$ and then $X_4X_5X_6X_7X_8X_9$. The flipped qubit has no effect here, since

$$X_4X_5X_6\left(\frac{|1\rangle|0\rangle|0\rangle+|0\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)=\left(\frac{|1\rangle|0\rangle|0\rangle+|0\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)$$

$$X_4X_5X_6\left(\frac{|1\rangle|0\rangle|0\rangle-|0\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)=-\left(\frac{|1\rangle|0\rangle|0\rangle-|0\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)$$

which is exactly what we'd expect from the second block if the fourth qubit hadn't been flipped. Therefore the results of the two observables are 1 and 1, so the phase-flip error syndrome is 0: no phase flip.

Finally, to fix the bit-flip error on the fourth qubit, we apply X_4.

$$X_4\left[\alpha\left(\cdots\right)\left(\frac{|1\rangle|0\rangle|0\rangle+|0\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\cdots\right)+\beta\left(\cdots\right)\left(\frac{|1\rangle|0\rangle|0\rangle-|0\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\cdots\right)\right]=|\psi_L\rangle$$

and recover the original state.

■

Example (fixing phase-flips) Inexplicably, the eighth qubit in $|\psi_L\rangle$ has been phase-flipped, leaving the logical qubit in the state

$$|\psi_L\rangle\longrightarrow\begin{cases}\alpha\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\\ +\beta\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\end{cases}$$

Once again, we don't know where/what/if the error is, so we'll start by checking for bit flips. Using the Z_1Z_2, Z_2Z_3, then Z_4Z_5, Z_5Z_6, then Z_7Z_8, Z_8Z_9 (because we already used the projection operators $\{P_j\}$ in the last example) we can check for bit-flip errors. Doing this sort of thing six times

$$\begin{aligned}&Z_1Z_2\left[\alpha\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)(\cdots)+\beta\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)(\cdots)\right]\\ =&\left[\alpha\left(\frac{|0\rangle|0\rangle|0\rangle+(-1)(-1)|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)(\cdots)+\beta\left(\frac{|0\rangle|0\rangle|0\rangle-(-1)(-1)|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)(\cdots)\right]\\ =&\left[\alpha\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)(\cdots)+\beta\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)(\cdots)\right]\end{aligned}$$

we find that there are no bit-flip errors.

Using $X_1X_2X_3X_4X_5X_6$ and then $X_4X_5X_6X_7X_8X_9$ we can check for phase flip errors.

$$
\begin{aligned}
& X_1X_2X_3X_4X_5X_6\left[\alpha\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)+\beta\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)\right] \\
= \; & \left[\alpha\left(\tfrac{|1\rangle|1\rangle|1\rangle+|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right)\left(\tfrac{|1\rangle|1\rangle|1\rangle+|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right)(\cdots)+\beta\left(\tfrac{|1\rangle|1\rangle|1\rangle-|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right)\left(\tfrac{|1\rangle|1\rangle|1\rangle-|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right)(\cdots)\right] \\
= \; & \left[\alpha\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)+\beta(-1)(-1)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)\right] \\
= \; & \left[\alpha\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)+\beta\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)(\cdots)\right]
\end{aligned}
$$

Blocks one and two have matching phase.

$$
\begin{aligned}
& X_4X_5X_6X_7X_8X_9\left[\alpha(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)+\beta(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\right] \\
= \; & \left[\alpha(\cdots)\left(\tfrac{|1\rangle|1\rangle|1\rangle+|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right)\left(\tfrac{|1\rangle|1\rangle|1\rangle-|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right)+\beta(\cdots)\left(\tfrac{|1\rangle|1\rangle|1\rangle-|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right)\left(\tfrac{|1\rangle|1\rangle|1\rangle+|0\rangle|0\rangle|0\rangle}{\sqrt{2}}\right)\right] \\
= \; & \left[\alpha(1)(-1)(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)+\beta(-1)(1)(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\right] \\
= \; & -\left[\alpha(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)+\beta(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\right]
\end{aligned}
$$

Blocks two and three do not have matching phase, so evidently the third block has experienced a phase-flip error.

In order to fix this phase-flip on the third block (it doesn't matter that it was specifically the eighth qubit) we apply $Z_7Z_8Z_9$.

$$
\begin{aligned}
& Z_7Z_8Z_9\left[\alpha(\cdots)(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)+\beta(\cdots)(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\right] \\
= \; & \left[\alpha(\cdots)(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle-(-1)(-1)(-1)|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)+\beta(\cdots)(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle+(-1)(-1)(-1)|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\right] \\
= \; & \left[\alpha(\cdots)(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)+\beta(\cdots)(\cdots)\left(\tfrac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\right] \\
= \; & |\psi_L\rangle
\end{aligned}
$$

■

Measurement as a Tool

Example In the last two examples we used measurements to get information about the state; to find out if there has been a bit or phase flip error. But measurements themselves have an impact on the state: they project the state into predetermined subspaces, the range of the detection operators $\{A_j\}$. Knowing the result of the measurement means we can intelligently control the result.

Say you'd like to produce lots of horizontally polarized light and all you've got is a completely random blackbody source (a light bulb). Maybe you're building half of a big 3D movie projector or something. You could just put a horizontal polarizing filter in front of your light source, but half of your light is immediately lost (to a rapidly heating polarizer).

Instead of a horizontal filter, which destroys the vertical photons, we could use a polarizing beam splitter (PBS) to reflect the vertical photons. On that second path we place a $\frac{\pi}{2}$ polarization rotator[4] to convert the vertical to horizontal photons. The only trace of the original state is in the intensity of light in the two beams, $|\langle 0|\psi\rangle|^2 = Tr\left[|0\rangle\langle 0|\rho\right]$ and $|\langle 1|\psi\rangle|^2$.

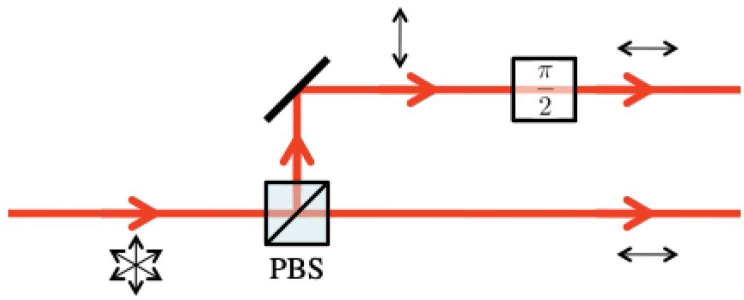

Figure 4: Making every photon horizontally polarized using "error correction". We measure the polarization in the $\{|0\rangle, |1\rangle\}$ basis using a polarizing beam splitter, reflecting vertical photons, $|0\rangle$, and transmitting horizontal photons, $|1\rangle$. "Correct" horizontal results are left alone and "error" vertical results are "repaired" using with a $\frac{\pi}{2}$ polarization rotation.

It may bother you that the horizontal light is on two channels. This is a good way to think about error correction; $\boxed{\text{the measurement yields a result that we remember}}$ and which governs how we repair it, so the final state isn't just "horizontal", it's "horizontal and you fixed it". The measurement result is an active part of the larger system, and in this case that result is "recorded" on the photon's "which path" information.

Example ("unmeasuring") Someone in the lab looked up at the wrong time and accidentally measured the first qubit. It's a $|1\rangle_1$.

The probability of that having happened is

$$p\left(|1\rangle_1\right) = |\langle 1_1|\psi_L\rangle|^2 = \frac{|\alpha|^2 + |\beta|^2}{2} = \frac{1}{2}$$

so the impact of this measurement is

[4]Sugar water rotates the polarization of light that passes through it, so this is not some profound, micro-controlled laboratory-only operations.

$$|\psi_L\rangle \xrightarrow{|1\rangle_1} |\psi_e\rangle = \sqrt{2}|1\rangle_1\langle 1|_1|\psi_L\rangle = \begin{cases} \alpha\left(|1\rangle|1\rangle|1\rangle\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \\ +\beta\left(-|1\rangle|1\rangle|1\rangle\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \end{cases}$$

Where $|\psi_e\rangle$ is the new "error state". As before, we begin by checking for bit-flip errors using the projection measurement, $\{P_j\}$, on each block. For the first block, the probability of each error syndrome is

$$\begin{aligned} p(0) &= \langle\psi_e|P_0|\psi_e\rangle = 1 \\ p(1) &= \langle\psi_e|P_1|\psi_e\rangle = 0 \\ p(2) &= \langle\psi_e|P_2|\psi_e\rangle = 0 \\ p(3) &= \langle\psi_e|P_3|\psi_e\rangle = 0 \end{aligned}$$

because $P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$. The bit-flip error syndrome is always zero, so we nothing to the state and $|\psi_e\rangle$ is passed on to the phase-flip check.

To check for phase-flip errors we use $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$. The latter yields a result of 1 (meaning that the phase of the second and third blocks is the same). The former does something interesting.

Since $X_1X_2X_3X_4X_5X_6 = X_1X_2X_3 \otimes X_4X_5X_6$, where

$$X_4X_5X_6\left(\frac{|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) = \left(\frac{|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)$$

$$X_4X_5X_6\left(\frac{|0\rangle|0\rangle|0\rangle - |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) = -\left(\frac{|0\rangle|0\rangle|0\rangle - |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)$$

and we already know exactly what $X_4X_5X_6$ is going to do with the intact second block in $|\psi_e\rangle$, we'll concentrate just on $X_1X_2X_3$ and the first block, $|111\rangle$.

The projection operators for the observable $X_1X_2X_3$ are

$$P^{(-1)} = |---\rangle\langle ---| + |++-\rangle\langle ++-| + |+-+\rangle\langle +-+| + |-++\rangle\langle -++|$$

$$P^{(1)} = |+++\rangle\langle +++| + |--+\rangle\langle --+| + |-+-\rangle\langle -+-| + |+--\rangle\langle +--|$$

since $|\langle +|1\rangle|^2 = |\langle -|1\rangle|^2$ all eight of these individual projections, and therefore the two measurement operators, are equally likely, so $p(1) = p(-1) = \frac{1}{2}$.

Again, because all eight of these individual projections are equally likely, the effects of these two measurements are:

$$\frac{P^{(1)}|111\rangle}{\sqrt{\frac{1}{2}}} = \frac{1}{2}|+++\rangle + \frac{1}{2}|--+\rangle + \frac{1}{2}|-+-\rangle + \frac{1}{2}|+--\rangle$$

$$= \begin{cases} \frac{1}{4\sqrt{2}}\left[|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\right] \\ +\frac{1}{4\sqrt{2}}\left[|000\rangle + |001\rangle - |010\rangle - |011\rangle - |100\rangle - |101\rangle + |110\rangle + |111\rangle\right] \\ +\frac{1}{4\sqrt{2}}\left[|000\rangle - |001\rangle + |010\rangle - |011\rangle - |100\rangle + |101\rangle - |110\rangle + |111\rangle\right] \\ +\frac{1}{4\sqrt{2}}\left[|000\rangle - |001\rangle - |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle\right] \end{cases}$$

$$= \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

$$\frac{P^{(-1)}|111\rangle}{\sqrt{\frac{1}{2}}} = -\frac{1}{2}|---\rangle - \frac{1}{2}|++-\rangle - \frac{1}{2}|+-+\rangle - \frac{1}{2}|-++\rangle$$

$$= \begin{cases} -\frac{1}{4\sqrt{2}}\left[|000\rangle - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle\right] \\ -\frac{1}{4\sqrt{2}}\left[|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle\right] \\ -\frac{1}{4\sqrt{2}}\left[|000\rangle + |001\rangle - |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle\right] \\ -\frac{1}{4\sqrt{2}}\left[|000\rangle + |001\rangle + |010\rangle + |011\rangle - |100\rangle - |101\rangle - |110\rangle - |111\rangle\right] \end{cases}$$

$$= \frac{-|000\rangle + |111\rangle}{\sqrt{2}}$$

So the possible results are $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$ or $-\frac{|000\rangle-|111\rangle}{\sqrt{2}}$. Remember that we're applying $X_1X_2X_3X_4X_5X_6 = X_1X_2X_3 \otimes X_4X_5X_6$, so the 1 result corresponds to $\left(\frac{|0\rangle|0\rangle|0\rangle \pm |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle \pm |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)$ and the -1 result corresponds to $\left(\frac{|0\rangle|0\rangle|0\rangle \pm |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle \mp |1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)$. In other words, the original "broken" state

$$|\psi_e\rangle = \begin{cases} \alpha\left(|1\rangle|1\rangle|1\rangle\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \\ +\beta\left(-|1\rangle|1\rangle|1\rangle\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \end{cases}$$

is "collapsed" by the phase-flip measurement into either

$$|\psi_e^{(1)}\rangle = \begin{cases} \alpha\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \\ +\beta\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \end{cases}$$

or

$$|\psi_e^{(-1)}\rangle = \begin{cases} -\alpha\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \\ -\beta\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \end{cases}$$

So, with a result of -1 we know that the phases of blocks one and two don't match and, combined with the result from $X_4X_5X_6X_7X_8X_9$, we know that block one has a flipped phase. Applying $Z_1Z_2Z_3$:

$$Z_1Z_2Z_3|\psi_e^{(-1)}\rangle = -|\psi_L\rangle$$

which (ignoring the global phase) is the original logical qubit restored!

We found that there were no bit-flips, and with a result of 1, we confirm that the phases of all three blocks match. Evidently there's nothing to do, and indeed a quick look at $|\psi_e^{(1)}\rangle$ verifies that.

$$|\psi_e^{(1)}\rangle = |\psi_L\rangle$$

■

This is remarkable. I mean seriously, this is a moment to turn off the music, sit down and ponder.

When we pass diagonally polarized light through a vertical/horizontal beam splitter, we're asking the photon "Are you vertical or horizontal?" and even though the photon isn't either (it's a superposition), we find that it always "answers" the question anyway, and will subsequently be found to be one or the other. Even though there are a continuous range of possible polarizations, the beam splitter finds that every photon is either vertical or horizontal every time. The same thing is happening here.

By measuring the error syndrome, we not only discover what the error is, but we force the error to take a form that we can manage. Even though we have to contend with an infinite, continuous set of errors, the mere act of asking "Bit-flip or phase-flip?" forces the corrupted state into one of a finite set of "fixable states", related to the logical qubit by no more than a simple bit-flip and/or phase-flip.

In this last example, we committed the greatest sin that can be done in quantum information: we observed a qubit ahead of schedule. But the Shor Code turned that error into either nothing half the time or a phase flip the other half of the time. A simple, easy to fix, and most important *standard* error.

It's like a guy going to the hospital with a broken foot, where the receptionist asks "Is your nose running?" He finds that his nose is in fact running, so the receptionist hands him a handkerchief from a pile and, with his problem solved, he leaves the hospital without ever needing to see a doctor.

Exercises

The "logical qubit" that represents $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the Shor Code is

$$|\psi_L\rangle = \begin{cases} \alpha\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \\ +\beta\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle}{\sqrt{2}}\right) \end{cases}$$

1) See Shor

The fifth qubit has suffered both a phase-flip error and a bit-flip error.

a) What is the state, $|\psi_e\rangle = X_5 Z_5 |\psi_L\rangle$, of the corrupted logical qubit?

b) Using the appropriate measurements and repair operations, check for bit-flip errors and fix any you find.

c) Using the appropriate measurements and repair operations, check for phase-flip errors and fix any you find.

You should have recovered $|\psi_L\rangle$ at the end of this step.

2) Not A Shor Thing

The seventh and ninth qubits have both suffered bit-flip errors.

a) What is the state, $|\psi_e\rangle = X_7 X_9 |\psi_L\rangle$, of the corrupted logical qubit?

b) Using the appropriate measurements and repair operations, check for bit-flip errors and fix any you find.

c) Using the appropriate measurements and repair operations, check for phase-flip errors and fix any you find.

d) What went wrong?

3) Shor-ly Not Another Shor Pun

Show that using the observables $Z_1 Z_2$ followed by $Z_2 Z_3$ is equivalent to using the four projection operators, $\{P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|, P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|, \ldots\}$, by direct calculation.

Remember that using the observable $Z_1 Z_2$ means using the projection operators corresponding to the eigenstates, $P^{(1)} = [|00\rangle\langle 00| + |11\rangle\langle 11|] \otimes I_3$, $P^{(-1)} = [|01\rangle\langle 01| + |10\rangle\langle 10|] \otimes I_3$.