QI Lecture 21

Coherence and The Quantum Eraser

The Set Up

Define A to be the "photon space", composed of N paths, and the state of a photon in path j as $|j\rangle_a \in A$. Define B to be the "detector space" with $|\eta_k\rangle_b \in B$ and $|\eta_0\rangle_b$ the state of the detector before interacting with the photons in system A. We'd like to investigate and *quantify* the coherence in A given a measurement by B.

The initial state of the system is

$$|\psi\rangle = \left(\sum_j \sqrt{p(j)}|j\rangle_a\right)|\eta_0\rangle_b$$

where $p(j)$ is the probability of the photon being found on path j. The interaction with the detector is given by

$$U|\psi\rangle = \sum_j \sqrt{p_j} U\left[|j\rangle_a|\eta_0\rangle_b\right] = \sum_j \sqrt{p_j}|j\rangle_a|\eta_j\rangle_b$$

This detector may not be perfect, so we we'll assume that in general $\langle\eta_j|\eta_k\rangle \neq \delta_{jk}$. The reduced density matrices are

$$\rho_a = \sum_{jk} \sqrt{p_j p_k}\langle\eta_j|\eta_k\rangle_b |k\rangle_a\langle j|_a \qquad \rho_b = \sum_j p_j|\eta_j\rangle_b\langle\eta_j|_b$$

Notice that if the detector states are orthogonal, then the detector states perfectly label the paths, and if the detector states are are equal, then the original state can be factored out (and is thus unaffected).

Quantifying Coherence

A good way to quantify the coherence of ρ with respect to some basis, $\{|k\rangle\}$, is the "**Relative Entropy Coherence Measure**"

$$\boxed{C[\rho] \equiv S[\rho_d] - S[\rho]}$$

where ρ_d is equal to ρ on its diagonal, and zero otherwise,

$$\boxed{\rho_d \equiv \sum_k |k\rangle\langle k|\rho|k\rangle\langle k|}$$

Example Find the coherence of a completely mixed qubit, a photon polarized at angle θ, and a Bell state.

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \sim \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\theta = |\theta\rangle\langle\theta| \sim \begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{bmatrix}$$

$$\xi = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) \sim \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

We easily find that

$$\rho_d = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \sim \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\theta_d = \cos^2(\theta)|0\rangle\langle 0| + \sin^2(\theta)|1\rangle\langle 1| \sim \begin{bmatrix} \cos^2(\theta) & 0 \\ 0 & \sin^2(\theta) \end{bmatrix}$$

$$\xi_d = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| \sim \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The entropy of the diagonal density matrices are just the Shannon entropy of their entries. θ and ξ are both pure states, and therefore

$$C[\rho] = S[\rho_d] - S[\rho] = 1 - 1 = 0$$

$$C[\theta] = S[\theta_d] - S[\eta] = H\left[\left\{\cos^2(\theta), \sin^2(\theta)\right\}\right] - 0 = H\left[\left\{\cos^2(\theta), \sin^2(\theta)\right\}\right]$$

$$C[\xi] = S[\xi_d] - S[\xi] = 1 - 0 = 1$$

Predictably, the classical mixed state, $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$, has zero coherence. $|\Phi_+\rangle$ is an even superposition of two basis states, with 1 bit[1] of coherence.

Coherence is a measure of the superposition of two basis states. Considering the state θ, we see that

$$C\left[0\right] = C\left[\frac{\pi}{2}\right] = 0$$

which corresponds to $|\theta\rangle = |0\rangle, |1\rangle$. The coherence is maximized for

$$C\left[\frac{\pi}{4}\right] = C\left[-\frac{\pi}{4}\right] = 1$$

corresponding to $|\theta\rangle = |\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$.

■

Quantifying Which-Path Information

Alice could use the paths to send information by explicitly choosing to send each photon via $|x\rangle$ with probability $p(x)$. We'll define the "which path information" as the mutual information between Alice and Bob. This is literally the amount of information that Bob has about which path the photon takes.

In order to determine which path the photon took, Bob uses a POVM, $\{\Pi_y\}$, on the detector states. We have that

$$p(y|x) = Tr\left[\Pi_y |\eta_x\rangle\langle\eta_x|\right]$$

$$p(x,y) = p(x)p(y|x) = p(x)Tr\left[\Pi_y |\eta_x\rangle\langle\eta_x|\right]$$

$$p(y) = \sum_x p(x,y) = \sum_x p(x)Tr\left[\Pi_y |\eta_x\rangle\langle\eta_x|\right] = Tr\left[\Pi_y\left(\sum_x p(x)|\eta_x\rangle\langle\eta_x|\right)\right] = Tr\left[\Pi_y \rho_b\right]$$

which is all we need to define the mutual information between the measurement results, Y, and the state preparation, X.

[1]Since we're using information theoretic measures, we may as well inherit the units.

$$I[X;Y] \equiv H[Y] - H[Y|X] = -\sum_y p(y)\log(p(y)) + \sum_x p(x)\left[\sum_y p(y|x)\log(p(y|x))\right]$$

or actually plugging everything in

$$\boxed{I[X;Y] \equiv H\left[\{Tr\left[\Pi_y \rho_b\right]\}_y\right] - \sum_x p(x) H\left[\{Tr\left[\Pi_y |\eta_x\rangle\langle\eta_x|\right]\}_y\right]}$$

where the probability distributions used in the Shannon entropies are indexed with y. We have just set up exactly the situation needed to apply Holevo's Bound (lecture 20).

$$I[X;Y] \leq S\left[\sum_x p(x)|\eta_x\rangle\langle\eta_x|\right] - \sum_x p(x) S[|\eta_x\rangle\langle\eta_x|] = S\left[\rho_b\right] - 0 = S\left[\rho_b\right]$$

Uncertainty In Coherence and Which Path Information

The coherence of the photons between the paths in A is

$$C\left[\rho_a\right] \equiv S\left[\sum_k \sqrt{p_k p_k}\langle\eta_k|\eta_k\rangle_b |k\rangle_a\langle k|_a\right] - S\left[\rho_a\right] = S\left[\sum_k p_k |k\rangle_a\langle k|_a\right] - S\left[\rho_a\right] = H\left[\{p_k\}\right] - S\left[\rho_a\right]$$

$H\left[\{p_k\}\right]$ is determined entirely by the pre-measurement photon state, $\sum_j \sqrt{p(j)}|j\rangle_a$. We'll call the the "Preparation Entropy", because it describes how spread out the state is over the N paths.

The state of the full system is

$$\rho = U|\psi\rangle\langle\psi|U^\dagger = \left(\sum_j \sqrt{p(j)}|j\rangle_a|\eta_j\rangle_b\right)\left(\sum_k \sqrt{p(k)}\langle k|_a\langle\eta_k|_b\right)$$

which is pure. Therefore, the entropy of the reduced densities matrices are equal to each other and the entanglement between the systems, $S[\rho_a] = S[\rho_b]$. Begining with the Holevo Bound above,

$$\begin{aligned} I[X;Y] &\leq S\left[\rho_b\right] \\ I[X;Y] &\leq S\left[\rho_a\right] \\ I[X;Y] &\leq H\left[\{p_k\}\right] - C\left[\rho_a\right] \\ C\left[\rho_a\right] + I[X;Y] &\leq H\left[\{p_k\}\right] \end{aligned}$$

This bound relates how much we can know about which state a quantum system is in and how coherent that state can be.

$$\boxed{\underbrace{C\left[\rho_a\right]}_{Coherence} + \underbrace{I[X;Y]}_{Which\,Path} \le \underbrace{H\left[\{p_k\}\right]}_{Preparation}}$$

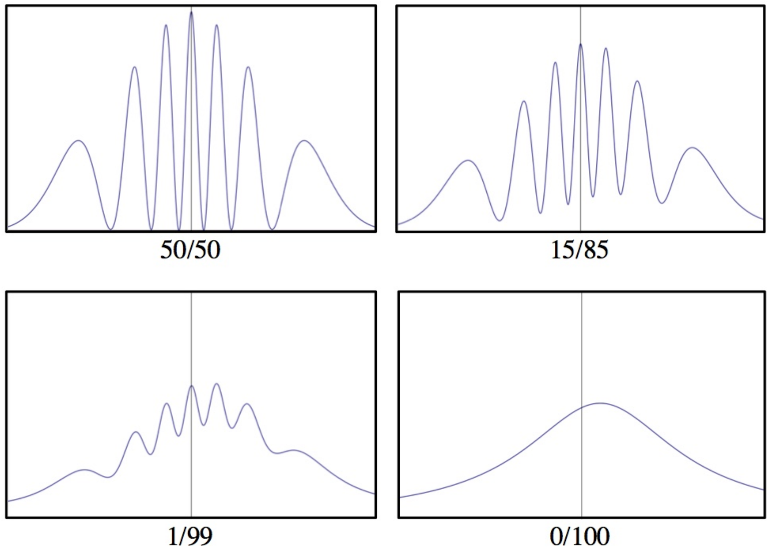

Figure 1: A very simple way to see the relationship between which-path information and coherence is to do the double slit with differently sized slits, so that you have some idea of which slit a photon went through (more likely the larger) and yet without complete path information you still have some coherence.

Example We'd like to extract some which-path information from $|+\rangle_a = \frac{1}{\sqrt{2}}|0\rangle_a + \frac{1}{\sqrt{2}}|1\rangle_a$ without completely destroying the coherence between $|0\rangle_a$ and $|1\rangle_a$. The "preparation entropy" here is $H\left[\left\{\frac{1}{2},\frac{1}{2}\right\}\right] = 1$.

We introduce a $|\theta\rangle_b = \cos(\theta)|0\rangle_b + \sin(\theta)|1\rangle_b$ ancilla qubit and then perform a CNOT to entangle them. Notice that if we choose $|\theta\rangle = |0\rangle$ we gain full path information, since

$$|+\rangle_a|0\rangle_b \xrightarrow{CNOT} \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

and if we choose $|\frac{\pi}{4}\rangle = |+\rangle$ we gain no path information, because

$$|+\rangle_a|+\rangle_b \xrightarrow{CNOT} |+\rangle_a|+\rangle_b$$

It will be more interesting to us to see something in between, such as $|\frac{\pi}{6}\rangle$.

$$|+\rangle_a|\frac{\pi}{6}\rangle_b = \left(\frac{|0\rangle_a + |1\rangle_a}{\sqrt{2}}\right)\left(\frac{\sqrt{3}|0\rangle_b + |1\rangle_b}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle$$

Performing CNOT to entangle them:

$$\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle + \frac{1}{2\sqrt{2}}|10\rangle$$

Let's write this using the frame work above.

$$|\psi\rangle = \sum_j \sqrt{p(j)}|j\rangle_a|\eta_j\rangle_b = \frac{1}{\sqrt{2}}|0\rangle_a\left(\frac{\sqrt{3}}{2}|0\rangle_b + \frac{1}{2}|1\rangle_b\right) + \frac{1}{\sqrt{2}}|1\rangle_a\left(\frac{1}{2}|0\rangle_b + \frac{\sqrt{3}}{2}|1\rangle_b\right)$$

So we have the path states, $|0\rangle_a$ and $|1\rangle_a$, as well as the detector states

$$|\eta_0\rangle = \frac{\sqrt{3}}{2}|0\rangle_b + \frac{1}{2}|1\rangle_b \qquad\qquad |\eta_1\rangle = \frac{1}{2}|0\rangle_b + \frac{\sqrt{3}}{2}|1\rangle_b$$

Now, by measuring B we can gain some (but not perfect) information about which path the photon is in. Let's calculate the coherence. First,

$$\begin{aligned}
\rho_a &= \textstyle\sum_{j,k=0}^{1} \sqrt{p(j)p(k)}\langle\eta_j|\eta_k\rangle_b|k\rangle_a\langle j|_a \\
&= \tfrac{1}{2}|0\rangle_a\langle 0|_a + \tfrac{\sqrt{3}}{4}|0\rangle_a\langle 1|_a + \tfrac{\sqrt{3}}{4}|1\rangle_a\langle 0|_a + \tfrac{1}{2}|1\rangle_a\langle 1|_a \\
&\sim \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix}
\end{aligned}$$

To find the entropy of this we need the eigenvalues.

$$\begin{aligned}
0 &= \begin{vmatrix} \frac{1}{2}-\lambda & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2}-\lambda \end{vmatrix} \\
0 &= \left(\tfrac{1}{2}-\lambda\right)^2 - \tfrac{3}{16} \\
0 &= \lambda^2 - \lambda + \tfrac{1}{4} - \tfrac{3}{16} \\
0 &= \lambda^2 - \lambda + \tfrac{1}{16} \\
\lambda &= \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{(-1)^2 - \tfrac{4}{16}} \\
\lambda &= \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{\tfrac{3}{4}} \\
\lambda &= \tfrac{1}{2} \pm \tfrac{\sqrt{3}}{4} \\
\lambda &\approx 0.067, 0.933
\end{aligned}$$

and therefore the coherence is

$$C[\rho_a] = S\left[\left[\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array}\right]\right] - S\left[\left[\begin{array}{cc} \frac{1}{2} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} \end{array}\right]\right] = 1 - H\left[\{0.067, 0.933\}\right] \approx 0.645\, bits$$

From the coherence and which-path bound we have that

$$\begin{aligned} C\left[\rho_a\right] + I[X;Y] &\leq H\left[\{p_k\}\right] \\ 0.645 + I[X;Y] &\leq 1 \\ I[X;Y] &\leq 0.355 \end{aligned}$$

On average, at most 0.355 bits of which-path information can be extracted from this state.

■

Quantum Eraser

In the quantum eraser experiment, we can turn a knob to switch between knowing or not knowing the which-path information from a double slit. A pair of polarization-entangled photons are produced in the Bell state

$$|\Psi_+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{i|R\rangle|R\rangle + i|L\rangle|L\rangle}{\sqrt{2}}$$

Where we've written the state in both the vertical/horizontal polarization basis, $\{|0\rangle, |1\rangle\}$, as well as the circular polarization basis, where

$$|\circlearrowleft\rangle = |L\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} \qquad\qquad |\circlearrowright\rangle = |R\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$

Typically, these kinds of entangled photons are produce via "spontaneous parametric down conversion", where a high frequency photon is spontaneously absorbed by an atom in a material and two new polarization-entangled photons photons are generated in two new directions.

Each photon in the pair is directed (eventually) two detectors D_A and D_B, which connect to a "**coincidence counter**". By disregarding events where only one detector clicks and keeping those where both click, we can select for those events (likely) generated by the entangled pair.

Scenario One: The B photon is sent through a pair of slits, producing the usual double slit interference pattern described as the sum of states coming out of each slit,

$$\frac{|\psi_1(x)\rangle + |\psi_2(x)\rangle}{\sqrt{2}}$$

where x describes the position at the "screen" where we hope to witness interesting interference patterns. The source of the B photon is small and the wavelength is well-defined, so this is like creating coherent light with a single slit before the double slit.[2]

We detect the interference patterns using detector D_B, which can move to various values of x. The entanglement of the photons' polarizations doesn't affect the interference pattern, because knowing about the polarization of A doesn't tell you anything about which slit B went through.

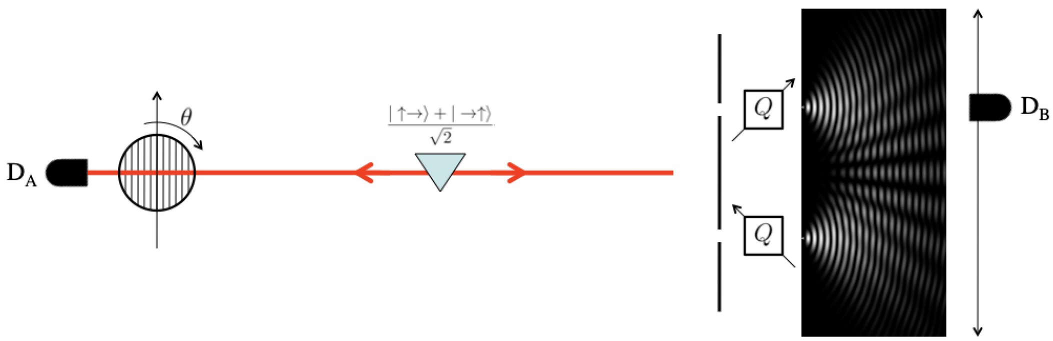

Figure 2: The set up for the quantum eraser.

Scenario Two: To mark the two slits, quarter wave plates,

$$Q\left(\frac{\pi}{4}\right) = i|+\rangle\langle+| + |-\rangle\langle-| \sim \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{-1+i}{2} & \frac{1+i}{2} \end{bmatrix} = \frac{1+i}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

$$Q\left(-\frac{\pi}{4}\right) = |+\rangle\langle+| + i|-\rangle\langle-| \sim \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix} = \frac{1+i}{2}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

aligned at $\theta = \pm\frac{\pi}{4}$ to the vertical, are positioned with one behind each slit. We can quickly calculate that,

[2]Which is how Young originally did it.

$$Q\left(\frac{\pi}{4}\right)|0\rangle \sim \frac{1+i}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1+i}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} \sim e^{i\frac{\pi}{4}}|L\rangle$$

$$Q\left(-\frac{\pi}{4}\right)|0\rangle \sim \frac{1+i}{2}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1+i}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} \sim e^{i\frac{\pi}{4}}|R\rangle$$

$$Q\left(\frac{\pi}{4}\right)|1\rangle \sim \frac{1+i}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1+i}{2}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{-1+i}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} \sim -e^{-i\frac{\pi}{4}}|R\rangle$$

$$Q\left(-\frac{\pi}{4}\right)|1\rangle \sim \frac{1+i}{2}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1+i}{2}\begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{1-i}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} \sim e^{-i\frac{\pi}{4}}|L\rangle$$

We've seen this before. When light polarized at an angle of 45^o to the "slow axis" of a QWP, it is converted to circularly polarized light. We can define the operator $\mathcal{Q}$ as the effect of both of these QWPs,

$$\begin{aligned}
\mathcal{Q}|0\rangle|\psi_1(x)\rangle &= e^{i\frac{\pi}{4}}|L\rangle|\psi_1(x)\rangle \\
\mathcal{Q}|0\rangle|\psi_2(x)\rangle &= e^{i\frac{\pi}{4}}|R\rangle|\psi_2(x)\rangle \\
\mathcal{Q}|1\rangle|\psi_1(x)\rangle &= -e^{-i\frac{\pi}{4}}|R\rangle|\psi_1(x)\rangle \\
\mathcal{Q}|1\rangle|\psi_2(x)\rangle &= e^{-i\frac{\pi}{4}}|L\rangle|\psi_2(x)\rangle
\end{aligned}$$

Defining C to be the which-path, interference generating component of photon B, the original state was

$$\left(\frac{|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b}{\sqrt{2}}\right)\left(\frac{|\psi_1\rangle_c + |\psi_2\rangle_c}{\sqrt{2}}\right)$$

which is separable. Therefore the which-path information is left alone and we see interference. But $\mathcal{Q}$ makes the which-path information available. Define the new state to be $|\eta\rangle$ to be the new state.

$$\begin{aligned}
& |\eta\rangle \\
= \; & \mathcal{Q}\left(\tfrac{|0\rangle_a|1\rangle_b+|1\rangle_a|0\rangle_b}{\sqrt{2}}\right)\left(\tfrac{|\psi_1\rangle_c+|\psi_2\rangle_c}{\sqrt{2}}\right) \\
= \; & -\tfrac{e^{-i\frac{\pi}{4}}}{2}|0\rangle_a|R\rangle_b|\psi_1\rangle_c + \tfrac{e^{-i\frac{\pi}{4}}}{2}|0\rangle_a|L\rangle_b|\psi_2\rangle_c + \tfrac{e^{i\frac{\pi}{4}}}{2}|1\rangle_a|L\rangle_b|\psi_1\rangle_c + \tfrac{e^{i\frac{\pi}{4}}}{2}|1\rangle_a|R\rangle_b|\psi_2\rangle_c \\
= \; & \tfrac{e^{-i\frac{\pi}{4}}}{2}|0\rangle_a\left[-|R\rangle_b|\psi_1\rangle_c + |L\rangle_b|\psi_2\rangle_c\right] + \tfrac{e^{i\frac{\pi}{4}}}{2}|1\rangle_a\left[|L\rangle_b|\psi_1\rangle_c + |R\rangle_b|\psi_2\rangle_c\right]
\end{aligned}$$

Tracing over A we see that the paths are clearly marked by their polarizations

$$Tr_a[|\eta\rangle\langle\eta|] = \begin{cases} \frac{1}{2}\left(\frac{|R\rangle_b|\psi_1\rangle_c - |L\rangle_b|\psi_2\rangle_c}{\sqrt{2}}\right)\left(\frac{\langle R|_b\langle\psi_1|_c - \langle L|_b\langle\psi_2|_c}{\sqrt{2}}\right) \\ +\frac{1}{2}\left(\frac{|L\rangle_b|\psi_1\rangle_c + |R\rangle_b|\psi_2\rangle_c}{\sqrt{2}}\right)\left(\frac{\langle L|_b\langle\psi_1|_c + \langle R|_b\langle\psi_2|_c}{\sqrt{2}}\right) \end{cases}$$

and tracing over A and B we see that the pattern on the screen should be the "classical" distribution, where there's no relationship between the two slits.

$$Tr_{ab}[|\eta\rangle\langle\eta|] = \frac{1}{2}|\psi_1\rangle_c\langle\psi_1|_c + \frac{1}{2}|\psi_2\rangle_c\langle\psi_2|_c$$

As long as the QWPs are in place, this result is the rule: two bumps, one for each slit, and no visible interference pattern.

Scenario Three: In the last scenario the double slit interference was lost because it was possible, in theory, to know which slit the photon went through. However, this which-slit information can be "erased". In this last scenario we place a polarizer in from of detector A that we can set at any angle, θ.

The coincidence detector allows us to efficiently do measurement projections; if the polarizer is horizontally aligned, then the only states that will be acknowledged by the coincidence detector will include $|1\rangle_a$. Aligning the polarizer at $\theta = 0$, the coincidence detector only includes states of the form

$$|0\rangle_a\left(\frac{|R\rangle_b|\psi_1\rangle_c - |L\rangle_b|\psi_2\rangle_c}{\sqrt{2}}\right)$$

and aligning it at $\theta = \frac{\pi}{2}$ we only note states of the form

$$|1\rangle_a\left(\frac{|L\rangle_b|\psi_1\rangle_c + |R\rangle_b|\psi_2\rangle_c}{\sqrt{2}}\right)$$

In other words, as we move D_B across a range of values of x, we find that it clicks coincidentally with D_A in a "two bump" (no interference) pattern.

However, if we align the polarizer in front of A at $\theta = \pm\frac{\pi}{4}$, things change dramatically. Because $|\Psi_+\rangle = \frac{|++\rangle - |--\rangle}{\sqrt{2}}$, when we register a $|\pm\rangle$ state at D_A, we know that photon B has the same polarization. Because the QWPs are aligned at $\pm\frac{\pi}{4}$, the eigenstates of both are $\{|+\rangle, |-\rangle\}$ ($\lambda = 1$ for one and $\lambda = i$ for the other).

We can very easily find that

$$Q\left(\frac{\pi}{4}\right)|+\rangle = i|+\rangle \qquad Q\left(-\frac{\pi}{4}\right)|+\rangle = |+\rangle$$

$$Q\left(\frac{\pi}{4}\right)|-\rangle = |-\rangle \qquad Q\left(-\frac{\pi}{4}\right)|-\rangle = i|-\rangle$$

and therefore

$$\begin{aligned}
\mathcal{Q}|+\rangle|\psi_1(x)\rangle &= i|+\rangle|\psi_1(x)\rangle \\
\mathcal{Q}|+\rangle|\psi_2(x)\rangle &= |+\rangle|\psi_2(x)\rangle \\
\mathcal{Q}|-\rangle|\psi_1(x)\rangle &= |-\rangle|\psi_1(x)\rangle \\
\mathcal{Q}|-\rangle|\psi_2(x)\rangle &= i|-\rangle|\psi_2(x)\rangle
\end{aligned}$$

Once again, the original state before the QWPs is

$$\left(\frac{|+\rangle_a|+\rangle_b - |-\rangle_a|-\rangle_b}{\sqrt{2}}\right)\left(\frac{|\psi_1\rangle_c + |\psi_2\rangle_c}{\sqrt{2}}\right)$$

and with the QWPs it's

$$\frac{i}{2}|+\rangle_a|+\rangle_b|\psi_1(x)\rangle_c + \frac{1}{2}|+\rangle_a|+\rangle_b|\psi_2(x)\rangle_c - \frac{1}{2}|-\rangle_a|-\rangle_b|\psi_1(x)\rangle_c - \frac{i}{2}|-\rangle_a|-\rangle_b|\psi_2(x)\rangle_c$$

$$= \frac{1}{\sqrt{2}}|+\rangle_a|+\rangle_b\left(\frac{i|\psi_1(x)\rangle_c + |\psi_2(x)\rangle_c}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}|-\rangle_a|-\rangle_b\left(\frac{|\psi_1(x)\rangle_c + i|\psi_2(x)\rangle_c}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}|+\rangle_a|+\rangle_b\left(\frac{i|\psi_1(x)\rangle_c + |\psi_2(x)\rangle_c}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}|-\rangle_a|-\rangle_b\left(\frac{i|\psi_1(x)\rangle_c - |\psi_2(x)\rangle_c}{\sqrt{2}}\right)$$

This means that when measure the pattern with D_B coincident with photon A being in the $|+\rangle$ state, we'll see interference! Doing the same conditioning on A being in the $|-\rangle$ state we see another, complimentary interference pattern!

These two states

$$\frac{i|\psi_1(x)\rangle_c + |\psi_2(x)\rangle_c}{\sqrt{2}} \qquad \frac{i|\psi_1(x)\rangle_c - |\psi_2(x)\rangle_c}{\sqrt{2}}$$

are complimentary in the sense that a bright spot in one will be a dark spot in the other (assuming that each path contributes about equally, which is fair), because where one pattern adds, the other subtracts.

Evidently, we can "erase" the which-path information. More precisely, we find that we can choose between having access to the which-path information and measuring the phase-difference information. The interference patterns don't magically show up or disappear when we start measuring A in a particular way; they're only visible when we keep track of coincidence.[3] Necessarily, the sum of these complimentary patterns is equal to the no-interference "two bumps" pattern.

Remarkably, we can turn the knob on the angle of the polarizer in front of D_A and smoothly move from a two-bump pattern to wave interference. For example, if $\theta = 0.001$ and D_A clicks, then the state is probably $|0\rangle$, meaning that we don't expect to see interference, but it's also slightly more likely to be $|+\rangle$ than $|-\rangle$, meaning that one interference pattern is brighter than the other.

Delayed Choice Quantum Eraser

It may seem possible that in the quantum eraser experiment a conscious observer, by choosing particular measurements for A, is imposing a new state on B. That's worrisome for a lot of reasons.

The delayed choice quantum eraser attacks that notion in two ways. First, the choice of which kind of measurement to do on A is made *after* D_B has clicked, so there's no way that A will directly influence the outcome of measurements on B.[4] Second, the decision about which measurement to make is fundamentally random; nobody needs to bother being conscious.

The essential details are the same: photon B takes two paths and creates (possibly) an interference pattern that can be measured with a movable detector, D_0. However, rather than using QWPs to entangle the polarization and which-path states, we start photons A and B with entangled which-path states, $|top\rangle$ and $|bot\rangle$.

If D_3 or D_4 click, then the which-path information is revealed. We see this in practice as a disappearance of the interference fringes for the coincident detections in D_0. However, if D_1 clicks, then there's no way to recover the which-path information; it has been erased. However, exactly as with the regular quantum eraser experiment, the measurements made by D_1 and D_2 determine the phase difference between the two paths giving rise to complimentary interference patterns (figure 4).

Lacking coincidence counting, the pattern detected by D_0 looks exactly like the pattern produced by a complete lack of interference: "two bumps". But with coincidence counting, where we gain classical information from the far side of an entangled pair, we find that

[3] Yet another example of classical information changing a density matrix.

[4] Assuming you believe in causality, which the past has made a pretty compelling case for. The future has yet to weigh in.

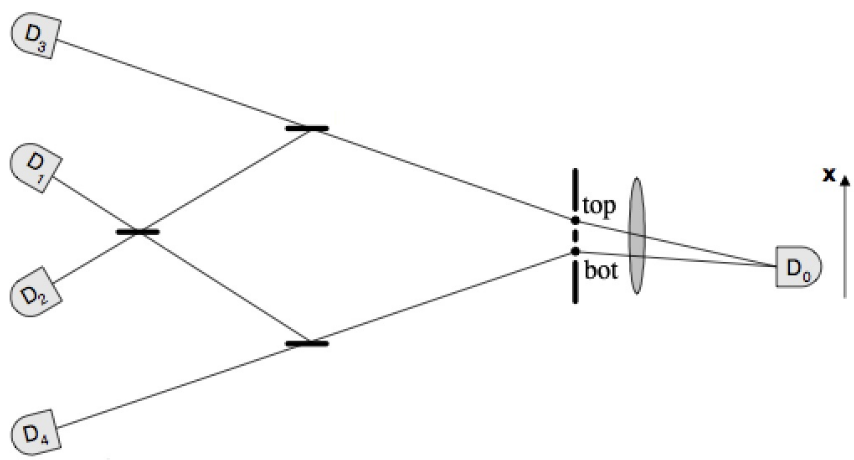

Figure 3: The delayed choice quantum eraser (somewhat idealized). The "slits" in this experiment are two atoms, which release a photon randomly. However, as with an actual double slit, the photons in this experiment come from a superposition of both sources. Because this isn't a real double slit with prepared coherent light, the phase relationship is induced through measurements (coincidence between D_0 and either D_1 or D_2).

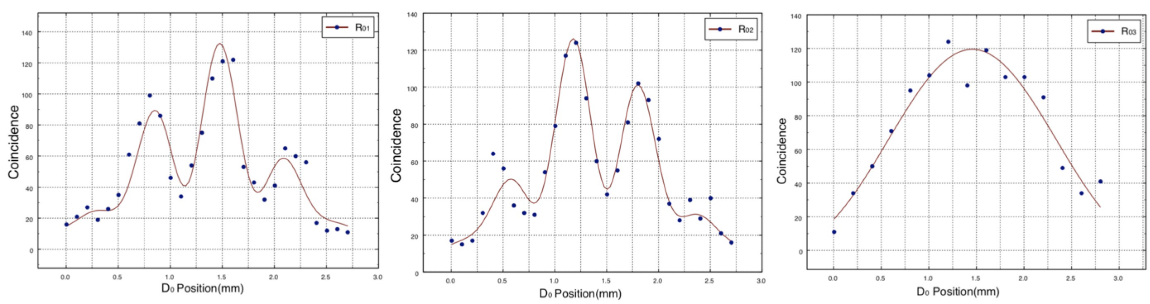

Figure 4: Empirical coincidence rates between D_0 and D_1 (left), D_2 (middle), and D_3 (right). Notice the complimentary interference patterns for D_1 and D_2 as well as the lack of interference for D_3.

there are patterns. Aggregating D_3 and D_4 we find that we can explain the two bumps as being generated by two sources of light. Simple enough. But aggregating D_1 and D_2 we can equally well explain the exact same two bump pattern as a being the overlap of two coherent interference patterns.

The same result, a particular photon arriving at a particular location to be detected by D_0 can be explained (consistently) in more than one way, depending on how its entangled twin is measured. Once again, the measurement of one photon has no direct effect of any kind on the pattern produced by the other (on its own, D_0 always measures two bumps), but the measurement can give us information that explains the pattern we see.

"Quantum Eraser" $\longrightarrow$ "Coherence Selector"

Notice that in the quantum eraser we don't simply "erase" the which-path information,

$$\boxed{\text{we actively select for a coherent state.}}$$

In the quantum eraser experiment, by positioning the polarizer at A to project using either P_0 or P_1, we make the which-path information available. A further measurement of the photon's polarization at the location of the screen would real both where it hits the screen as well as which slit it had gone through.

By positioning the A polarizer to project onto either P_+ or P_-, we select for either of the coherent states

$$\frac{i|\psi_1(x)\rangle_c + |\psi_2(x)\rangle_c}{\sqrt{2}} \qquad \frac{i|\psi_1(x)\rangle_c - |\psi_2(x)\rangle_c}{\sqrt{2}}$$

As far as quantum computation is concerned, the important thing about the delayed choice quantum eraser is that it projects with equal probability onto four path states, two with maximum coherence and two with maximum which-path information:

$$D_1 : \frac{|\psi_1\rangle + |\psi_2\rangle}{\sqrt{2}} \qquad D_2 : \frac{|\psi_1\rangle - |\psi_2\rangle}{\sqrt{2}} \qquad D_3 : |\psi_1\rangle \qquad D_4 : |\psi_2\rangle$$

The properties of these two states make sense in the context of the coherence and which-path bound. The states selected by D_1 and D_2 have maximum coherence, but zero which-path information. The states selected by D_3 and D_4 have zero coherence and complete which-path information.

It may bother you that we seem to be choosing between two different measurements. The delayed choice quantum eraser is more accurately be expressed as a POVM:

$$\begin{aligned}
\Pi_1 &= \tfrac{1}{4}\left(\tfrac{|\psi_1\rangle+|\psi_2\rangle}{\sqrt{2}}\right)\left(\tfrac{\langle\psi_1|+\langle\psi_2|}{\sqrt{2}}\right) \\
\Pi_2 &= \tfrac{1}{4}\left(\tfrac{|\psi_1\rangle-|\psi_2\rangle}{\sqrt{2}}\right)\left(\tfrac{\langle\psi_1|-\langle\psi_2|}{\sqrt{2}}\right) \\
\Pi_3 &= \tfrac{1}{4}|\psi_1\rangle\langle\psi_1| \\
\Pi_4 &= \tfrac{1}{4}|\psi_2\rangle\langle\psi_2|
\end{aligned}$$

But how is it possible to

Exercises

1) Dialing For Patterns

A $e^{i\theta}$ phase delay has been inserted into the delayed choice quantum eraser experiment as shown in figure 5. This question is mostly conceptual, so if you're doing a lot of math, you may be working too hard.

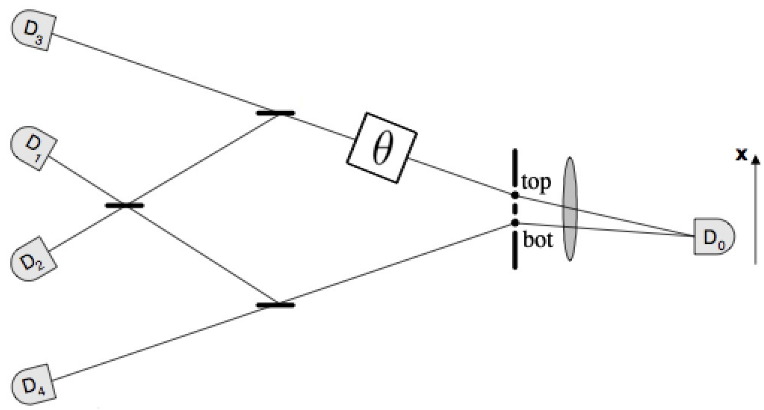

Figure 5: The DCQE with an extra phase added to one path.

a) Describe in words the effect of the phase delay on the coincidence patterns between D_0 with each of D_1, D_2, and D_3.

b) As you increase the phase delay, θ, do the individual interference fringes (coincident with either D_1 or D_2) physically move up or down? Explain your reasoning.

2) Which-Math Information

Here you'll be looking at the coherence/which-path relation:

$$\underbrace{C\left[\rho_a\right]}_{Coherence} + \underbrace{I[X;Y]}_{Which\,Path} \leq \underbrace{H\left[\{p_k\}\right]}_{Preparation}$$

Bob wants to measure the state $|-\rangle_a = \frac{|0\rangle_a - |1\rangle_a}{\sqrt{2}}$. After entangling it with a second system, he has the state

$$|\psi\rangle = \frac{|0\rangle_a|+\rangle_b - |1\rangle_a|1\rangle_b}{\sqrt{2}}$$

To measure this, Bob chooses a very simple POVM; a measurement in the computational basis.

$$\Pi_0 = |0\rangle_b\langle 0|_b \qquad \Pi_1 = |1\rangle_b\langle 1|_b$$

a) What is the preparation entropy for $|-\rangle_a$?

b) Define $\rho = |\psi\rangle\langle\psi|$. What is the coherence of ρ_a?

c) What is the mutual information between the state and Bob's measurement results?

d) Verify that the coherence/which-path inequality holds.

e) How can Bob alter his POVM to increase the coherence of ρ_a?

3) Coherence Adherence

$$|\psi\rangle = \frac{|0\rangle_a|?\rangle_b + |1\rangle_a|?\rangle_b}{\sqrt{2}} \qquad \rho = |\psi\rangle\langle\psi|$$

a) Find states for B that will minimize the coherence of ρ_a. What pair of states, when measured, would immediately reveal the state of A?

b) Find states for B that will maximize the coherence of ρ_a. What pair of states, when measured, would reveal nothing about the state of A?